What is Category theory?

- One level more abstract than other pure maths. "Maths about maths".
- A language for maths
- A way of thinking: we look at things in such a way that things look obvious.

We aren't just interested in one particular obj. (e.g. groups, rings...) but in how similar objects interact, in global structures and connections.

I. CATEGORIES, FUNCTORS & NATURAL TRANSFORMATIONS

A. Categories

A category \( \mathcal{C} \) consists of:
- a collection \( \text{ob} \mathcal{C} \) of objects (denoted \( A, B, C, \ldots \))
- for each pair \( A, B \in \text{ob} \mathcal{C} \) a collection \( \mathcal{C}(A, B) = \text{Hom}_\mathcal{C}(A, B) \) of morphisms (denoted \( f: A \to B, g, h, \ldots \))

equipped with:
- for each \( A \in \text{ob} \mathcal{C} \) an identity morphism \( \text{id}_A = 1_A \in \mathcal{C}(A, A) \)
- for each \( A, B, C \in \text{ob} \mathcal{C} \) a composition law \( \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C) \)
\[(f, g) \mapsto g \circ f = gf\]

Satisfying:

- **identity axioms** if \( f : A \to B \)
  
  \[1_B \circ f = f = f \circ 1_A\]

- **associativity** if \( f : A \to B, \; g : B \to C \)
  
  \[h : C \to D \quad \text{then} \quad (h \circ g) \circ f = h \circ (g \circ f)\]

A category \( \mathcal{C} \) is said to be **small** if \( \text{ob} \, \mathcal{C} \) and all of the \( \mathcal{C}(A, B) \) are sets and **locally small** if each \( \mathcal{C}(A, B) \) is a set.

**Remarks:**

- If \( f : A \to B \) we call \( A \) **domain** or **source** and \( B \) **codomain** or **target**.
- Morphisms are also referred to as **maps** or **arrows**.
- We won't worry too much about the intricacies of set theory.
- We could define a category by just considering morphisms, but in most examples the objects "come first."
- We may write \( \text{mor} \, \mathcal{C} \) for the collection of morphisms in \( \mathcal{C} \), and

\[
\begin{align*}
\text{dom} , \text{cod} : & \text{mor} \, \mathcal{C} \to \text{ob} \, \mathcal{C} \\
& \text{for the domain and codomain operations.}
\end{align*}
\]
Def. We say a square such as
\[
\begin{array}{ccc}
A & f & B \\
\downarrow & & \downarrow \\
D & g & C \\
\downarrow & & \downarrow \\
K & h & E
\end{array}
\]
commutes when the composites \( gf \) and \( kh \) give the same morphism \( A \to D \).

Examples:
(a) set of sets and functions
(b) categories of algebraic structures such as:
- \( \text{Gp} \) groups and group homomorphism
- \( \text{AbGp} \) abelian " " al.
- \( \text{Ring} \) rings " " ring
- \( \text{R-Mod} \) \( r \)-modules " " \( r \)-module

(c) categories of topological structures such as:
- \( \text{Top} \) : topological spaces and continuous maps
- \( \text{Haus} \) : Hausdorff spaces " " "
- \( \text{Met} \) : metric spaces and uniformly " " "
- \( \text{Htpy} \) : topological spaces and homotopy classes of continuous maps.

Def. A category with only identities as morphisms is called discrete.

Examples:
(d) mathematical structures viewed as categories
- Sets: any set can be viewed as a discrete category with the elements as objects.
- Posets: a poset $(P, \leq)$ can be regarded as a category with the elements of $P$ as objects, and $\text{Hom}(a, b)$ being a singleton if $a \leq b$ and empty otherwise. Then reflexivity $\Rightarrow$ existence of identity morphisms and transitivity $\Rightarrow$ composition.

Any category in which there is at most one morphism between any two objects is a preorder.

- Monoids: a locally small category with just one object is a monoid: the morphisms are the elements of the monoid, composition of morphisms is the multiplication in the monoid and the identity morphism is the unit of multiplication.

- Groups: a group can be considered as a category with one object just as the monoids. Now every morphism has a two-sided inverse.

**Def:** A morphism $f: A \to B$ in $C$ is an **isomorphism** if it has a two-sided inverse, i.e., a $g: B \to A$ satisfying $gf = 1_A$ and $fg = 1_B$.

A category in which every morphism is an isomorphism is called a groupoid.
Examples:  
- \( \text{iso } C \): any cat. gives rise to a groupoid; take all objects and all isomorphisms.

- Fundamental groupoid: given a space \( X \), the fundamental groupoid \( \Pi(X) \) has as objects the points of \( X \) and morphisms \( x \rightarrow y \) are homotopy classes of continuous paths \( u: [0,1] \rightarrow X \) from \( x \) to \( y \).

Composition of \( u: x \rightarrow y \) and \( v: y \rightarrow z \):

\[
u v(t) = \begin{cases} 
  u(2t) & 0 \leq t \leq 1/2 \\
  v(2t-1) & 1/2 \leq t \leq 1 
\end{cases}
\]

The identity morphism is the constant path at \( x \). Inverses \( u^{-1}(t) = u(1-t) \)
Examples ("New from old")

(a) Given cat $\mathcal{C}$, the opposite category $\mathcal{C}^\text{op}$ has the same objects, but the direction of the morphisms is reversed. $\mathcal{C}^\text{op}(A,B) = \mathcal{C}(B,A)$. This gives a "duality principle".

If some statement $P$ holds in any cat, then so does the statement $P^*$ obtained by "reversing all arrows" in $P$.

(b) Subcategories $\mathcal{D}$ is a subcat of $\mathcal{C}$ if

$\text{ob } \mathcal{D} \subseteq \text{ob } \mathcal{C}$ and $\forall a,b \in \text{ob } \mathcal{D}, \mathcal{D}(a,b) \subseteq \mathcal{C}(a,b)$

E.g. $A \text{Gp} \subseteq \text{Gp}$. $\subseteq \mathcal{C}(A,B)$

(c) Product categories. Given cats $\mathcal{C}$ and $\mathcal{D}$, the product $\mathcal{C} \times \mathcal{D}$ has objects $(A,B)$ with $A \in \text{ob } \mathcal{C}$ and $B \in \text{ob } \mathcal{D}$ and morphisms $(f,g):(A,B) \rightarrow (C,D)$ with $f:A \rightarrow C$ in $\mathcal{C}$ and $g:B \rightarrow D$ in $\mathcal{D}$

(d) Slice categories. Given a cat $\mathcal{C}$ and an object $B \in \text{ob } \mathcal{C}$, the slice category $\mathcal{C}/B$ has as objects those morphisms in $\mathcal{C}$ with codomain $B$. Morphisms are commutative triangles

$h: \left( \begin{array}{c} A \\ B \\ \hline f \\ g \end{array} \right) \rightarrow \left( \begin{array}{c} A \\ B \\ \hline f \end{array} \right)$ satisfies $A \xrightarrow{h} C$
Dually we have the coslice category
\[ B \downarrow C = (C^\text{op}/B)^\text{op} \]
where \( B \rightarrow C \).

* Eg: \( \text{Set}/B \) can be regarded as the category of "\( B \)-indexed families of sets".

An obj. \( (b, f^B_b) \) may be identified with the family \( (f^B_b) \mid b \in B \).

* \( 1 \downarrow \text{Set} \) (with \( 1 = \{\ast\} \) a one-pointed set)
  is the category of pointed sets; object are pairs \((A,a)\) of sets with a distinguished element \( a \in A \), and morphisms \( f: (A,a) \rightarrow (B,b) \)
  must preserve this: \( f(a) = b \).

* (e) **Arrow categories**: Given cat \( C \), the arrow cat \( \text{Arr}(C) \) has as objects the morphisms of \( C \)
  and as morphisms commutative squares:
  \[
  \begin{array}{ccc}
  A & \xrightarrow{f} & B \\
  \downarrow{g} & & \downarrow{h} \\
  C & \xrightarrow{k} & D
  \end{array}
  \]

(f) **Quotient categories**: Given an equivalence relation \( \sim \) on each collection of morphisms \( C(A,B) \) of a cat \( C \), satisfying
\[ f \sim g \Rightarrow fh \sim gh \quad \text{and} \quad kf \sim kg \]
whenever these composites are defined, then we can form the quotient cat \( C/\sim \).
Examples ("unusual maps")

- Matrices. Given a field $K$, let $\text{Mat}_K$ be the cat with nat. numbers as objects and $\text{Mat}_K(n,m)$ being num matrices with entries in $K$. The composition is matrix mult.

- Relations. $\text{Rel}$ is the cat which has sets as objects and morphisms $A \rightarrow B$ are triples $(A, R, B)$ where $R \subseteq A \times B$ is an arbitrary subset (a relation on $A \times B$).
  Composition of $(A, R, B)$ and $(B, S, C)$ is $(A, S \circ R, C)$ with
  $$S \circ R = \{(a, c) \mid \exists b \in B \ (a, b) \in R \land (b, c) \in S\}$$

- Partial functions. $\text{Part}$ has sets as objects and partial functions as morphisms. You can view a partial function as a relation $R \subseteq A \times B$ satisfying $(a, b) \in R \land (a, b') \in R \Rightarrow b = b'$.

- Formal proofs. Form a cat $\text{Proofs}$ with objects being logical statements (in some language) and morphisms being formal proofs of one statement from another (in a given logical system).
Examples ("finite categories")

(a) A discrete cat with 2 (or n) obj: \( j \to j' \)
(b) A cat with one non-identity morph: \( j \to j' \)
(c) A cat with two non-id morph: \( j \to j' \)
(d) \( \begin{array}{c}
\downarrow \\
\end{array} \)

B. Functors

Def Let \( \mathcal{C} \) and \( \mathcal{D} \) be cats

A functor \( \mathcal{F} : \mathcal{C} \to \mathcal{D} \) consists of:

- a mapping \( A \to \mathcal{F}A : \text{ob} \mathcal{C} \to \text{ob} \mathcal{D} \)
- mappings \( f \to \mathcal{F}f : \mathcal{C}(A,B) \to \mathcal{D}(\mathcal{F}A,\mathcal{F}B) \)

such that \( \mathcal{F}1_A = 1_{\mathcal{F}A} \)
\( \mathcal{F}(gf) = \mathcal{F}(g)\mathcal{F}(f) \) whenever \( gf \) is defined.

Examples

(a) Any cat \( \mathcal{C} \) has an identity functor.

We can also compose functions. This allows us to form the cat \( \text{Cat} \) of small cats and functors between them.

(b) If \( \mathcal{D} \) is a subcat of \( \mathcal{C} \) there is an inclusion functor \( \mathcal{D} \subset \mathcal{C} \).

If \( \mathcal{A} \times \mathcal{B} \) is a product cat, there are projection functors \( \mathcal{A} \times \mathcal{B} \to \mathcal{A} \) and \( \mathcal{A} \times \mathcal{B} \to \mathcal{B} \).
(c) **Forgetful functors**. We can define a functor

\[ U : \text{Gp} \to \text{Set} \]

which sends a group to its underlying set and a homomorphism to its underlying function. It "forgets" the group structure.

Similarly there are forgetful functors:

- \( \text{Rng} \to \text{Set} \)
- \( \text{R-Mod} \to \text{Set} \)
- \( \text{Top} \to \text{Set} \)
- \( \text{Rng} \to \text{Gp} \) (forgetting multiplication)

*(d) Free functor*. For any set \( A \), we can form the free group \( \text{FA} \) generated by \( A \).

Any function \( f : A \to B \) induces a unique group homomorphism \( \overline{f} : \text{FA} \to \text{FB} \) which sends any \( a \in A \) to \( f(a) \) in \( B \).

Given also \( g : B \to C \) we see that \( \overline{gf} = \overline{g} \circ \overline{f} \) as they agree on generators of \( \text{FA} \).

This gives a functor \( F : \text{Set} \to \text{Gp} \)

(e) There is a functor \( \text{Set} \to \text{Top} \) sending a set \( X \) to the discrete space on \( X \).

* (f) The Abelianisation functor \( \text{ab} : \text{Gp} \to \text{AbGp} \) sending \( G \) to \( G / [G, G] \)
(g) **Power set functor**: Define $P : \text{Set} \to \text{Set}$ by setting $P(A)$ to be the set of all subsets of $A$, and if $f : A \to B$ then $Pf(A) = \{ b \in B \mid (\exists a \in A') (b = f(a)) \} = f(A')$

We can also make the power set operation into a functor

$$P^* : \text{Set} \to \text{Set}^{\text{op}} \quad \text{(or Set}^{\text{op}} \to \text{Set})$$

by setting $P^* f(B) = f^{-1}(B)$

Check that $P^*(fg) = (P^*g)(P^*f)$

**Definition**: A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor $\mathcal{C}^{\text{op}} \to \mathcal{D}$ (or $\mathcal{C} \to \mathcal{D}^{\text{op}}$).

A functor which does not reverse the direction of arrows is also called covariant.

**Example**: (h) **Duals**: given a field $R$, we can form a functor $(-)^* : \text{k-Med} \to \text{k-Med}^{\text{op}}$ by sending a vector space $V$ to its dual $V^*$, and a linear map $f : V \to W$ to $f^* : W^* \to V^*$ which sends a linear functional $\phi \in W^*$ to $\phi f \in V^*$

Similarly there's a functor $(-)^* : \text{Rel} \to \text{Rel}^{\text{op}}$ defined by $A^* = A^T$ and $R^* = \{ (b,a) \mid (a,b) \in R \}$

(i) We can regard the operation $\mathcal{C} \to \mathcal{C}^{\text{op}}$ as a functor $\text{Cat} \to \text{Cat}$. If $F$ is a functor $F : \mathcal{C} \to \mathcal{D}$, then $F^{\text{op}}$ denotes the same data regarded as a functor $\mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$. Note that this is a covariant functor.
(j) A functor between monoids is a monoid homomorphism.

(k) A functor between partially ordered sets is an order-preserving map.

*(l) \textbf{Hom-Functor}: Given a locally small category \( \mathcal{C} \), there is for each object \( A \) of \( \mathcal{C} \) a hom-functor \( \mathcal{C}(A, -) : \mathcal{C} \rightarrow \text{Set} \).

\( \mathcal{C}(A, -) \) takes an object \( B \) to the set \( \mathcal{C}(A, B) \).

\( \mathcal{C}(A, -) \) applied to \( g : C \rightarrow D \) gives "postcomposition with \( g \)."

\( \mathcal{C}(A, g) : \mathcal{C}(A, C) \rightarrow \mathcal{C}(A, D) \)

\( f : A \rightarrow C \rightarrow A \xrightarrow{f} C \xrightarrow{g} D \)

Similarly, we have a contravariant hom-functor \( \mathcal{C}(-, A) : \mathcal{C}^{\text{op}} \rightarrow \text{Set} \).

(m) Let \( G \) be a group. Consider \( G \) as a cat with one object \(*). What is a functor \( G \rightarrow \text{Set} \)?

We have a set \( F(*) = A \) and for each \( g \in G \) a function \( \overline{g} = Fg : A \rightarrow A \), satisfying \( I = \overline{1}_A \), \( \overline{gh} = \overline{gh} \). This forces \( \overline{g}^{-1} = (\overline{g})^{-1} \).

So all \( \overline{g} \) are bijections.

So \( F \) is a \textbf{permutation representation}, or action of \( G \) on the set \( A \).
Similarly, for a given field $K$, functors $\mathcal{F} \to K\text{-Mod}$ are $K$-linear representations of $\mathcal{F}$.

(a) The fundamental group of a space defines a functor $\pi_1 : (\mathcal{C} \setminus \text{Top}) \to G\mathcal{P}$

(in fact, $(\mathcal{C} \setminus \text{Top}) \to G\mathcal{P}$, where $\simeq$ is basepoint-preserving homotopy).

The homology groups define functors $H_n : \text{Top} / \simeq \to G\mathcal{P}$

(in fact $\text{Top} / \simeq \to A_\mathbb{Z} G\mathcal{P}$)

Remark: Functors preserve commutative diagrams, so also properties defined by comm. diag's. e.g. isomorphisms

C. Natural Transformations

Natural transformations give a way of moving between the images of two functors.

Def. Let $\mathcal{C}, \mathcal{D}$ be cats and $F, G : \mathcal{C} \to \mathcal{D}$ two functors.

A natural transformation $\alpha$ from $F$ to $G$ is a collection of morphisms in $\mathcal{D}$

$\{ \alpha_A : FA \to GA \mid A \in \text{ob} \mathcal{C} \}$

satisfying $Gf \cdot \alpha_A = \alpha_B \cdot Ff \quad \forall f : A \to B \in \mathcal{C}$

\[ \begin{array}{ccc}
FA & \xrightarrow{\alpha_A} & GA \\
Ff \downarrow & & \downarrow Gf \\
FB & \rightarrow & GB
\end{array} \]

\textit{Naturality conditions}
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If \( \beta : G \rightarrow H \) is another natural transformation, then the composite \( \beta \alpha \) (given by \( (\beta \alpha)_A = \beta_A \alpha_A \)) is also a natural transformation.

For every functor \( F \) identity nat. transformation \( 1_F : F \rightarrow F \). So, given 2 cats \( C, D \), we have a functor category \([C,D]\):

the objects are functors from \( C \rightarrow D \), the morphisms are natural transformations between them.

Note that \([C,D](F,G) = \text{Nat}(F,G)\) is the class of natural transformations from \( F \) to \( G \).

If each \( \alpha_A \) is an isomorphism in \( D \), then we have another natural transformation \( G \rightarrow F \) given by \( \{ \alpha_A^{-1} : GA \rightarrow FA \} \) since \( (Ff)\alpha_A^{-1} = \alpha_F^{-1}\alpha_A^{-1}(Ff)\alpha_A^{-1} = \alpha_F^{-1}(Gf)\alpha_A^{-1} = \alpha_F^{-1}(Gf) \)

This makes \( \alpha \) an isomorphism in \([C,D]\), and we call it a natural isomorphism.

Examples  (a) For any vector space \( V \) we have a "natural" mapping \( \alpha_V : V \rightarrow V^{**} \) sending a vector \( v \in V \) to \( \psi \in \Psi(v) \)

This is the V-component of a nat. trans.

\[ 1_{\text{mod}} \rightarrow (-)^{**} \] i.e. for any linear map \( f : V \rightarrow W \) the diagram \( V \xrightarrow{\alpha} V^{**} \)

\[ f \downarrow \quad \alpha_W \downarrow \quad \text{Comutes.} \]
(b) Recall the covariant powerset functor: \( P : \text{Set} \rightarrow \text{Set} \).
For each set \( A \) let \( \{ \{ \cdot \} \}_a : A \rightarrow PA \) be the function
\( a \mapsto \{a\} \). Then \( \{ \} \) is a nat. trans. \( 1_{\text{Set}} \rightarrow P \).

(c) Let \( G, H \) be groups and \( f, g : G \rightarrow H \) group homs.
A nat. transf. \( \kappa : f \rightarrow g \) consists of an element \( C = \kappa_x \in H \) such that, for any \( x \in G \), we have
\[
\begin{array}{ccc}
  \ast & \xrightarrow{c} & \ast \\
  f(x) & \downarrow & g(x) \\
  \ast & \xrightarrow{c} & \ast
\end{array}
\]
i.e. \( g(x) = C f(x) C^{-1} \).
So \( \kappa \) is a conjugacy btw \( f, g \).

(d) The Hurewitz homomorphism \( h : \Pi_n(X, x) \rightarrow \Pi_n(X) \)
is a nat. transf. \( \Pi_n \rightarrow I \Pi_n \),
where \( I : (1 \setminus \text{Top}) / \sim \rightarrow \text{Top} / \sim \) forgets the base point,
and \( I : AbGP \rightarrow GP \) is the inclusion.
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YouTube: The Catsters by Eugenia Cheng.

D. Equivalences

Definition (a) We say \( F: \mathcal{C} \to \mathcal{D} \) is faithful if, for each \( f: A \to B \) in \( \mathcal{C} \), the equation \( Ff = Fg \) implies \( f = g \). i.e. "\( F \)", \( \mathcal{C}(A,B) \to \mathcal{D}(F(A),F(B)) \) is injective.

(b) We say \( F \) is full if for all objects \( A, B \) of \( \mathcal{C} \) and morphisms \( h: FA \to FB \) in \( \mathcal{D} \), \( \exists f: A \to B \) in \( \mathcal{C} \) with \( Ff = h \). i.e. \( \mathcal{C}(A,B) \to \mathcal{D}(FA,FB) \) is surjective.

(c) We say \( F \) is essentially surjective on objects if for every \( B \in \text{ob}\mathcal{D} \exists A \in \text{ob}\mathcal{C} \) with \( FA \cong B \).

(d) We say a subcategory \( \mathcal{C}' \) of \( \mathcal{C} \) is full if the inclusion functor \( \mathcal{C}' \to \mathcal{C} \) is a full functor.

For example, \( \text{Grp} \) is a full subcat of \( \text{Mon} \), but \( \text{Mon} \) is not a full subcat of \( \text{Semigroups} \) (monoids must necessarily have unit).

Definition Let \( \mathcal{C} \) and \( \mathcal{D} \) be cats. An equivalence btw \( \mathcal{C} \) and \( \mathcal{D} \) is a pair of functors \( F: \mathcal{C} \to \mathcal{D} \) and \( G: \mathcal{D} \to \mathcal{C} \) together with a pair of natural isomorphisms \( \alpha: 1_{\mathcal{C}} \to GF \) and \( \beta: 1_{\mathcal{D}} \to FG \).

We say \( \mathcal{C} \) and \( \mathcal{D} \) are equivalent, \( \mathcal{C} \cong \mathcal{D} \) if there is an equivalence between them.
Lemma ("equivalence $\iff$ full, faithful")

Let $F : C \to D$

(i) If $F$ is part of an equivalence $(F, G, \alpha, \beta)$ then $F$ is full, faithful and essentially surjective on objects.

(ii) The converse holds if we assume a "sufficiently big" axiom of choice.

Proof: (i) $F$ is faithful for any $f : A \to B$ in $C$, we can recover $f$ from $Ff$.

\[
\begin{align*}
A & \xrightarrow{f} B \\
\alpha_A & \cong \alpha_B \\
GFA & \xrightarrow{FF} GFB
\end{align*}
\]

So $f = \alpha_B^{-1} GFF \alpha_A$.

So $FF = Fg$ implies $f = g$.

(Or course $G$ is also faithful.)

$F$ is full.

Given $h : FA \to FB$, define

\[
\begin{align*}
A & \to B \\
\alpha_A & \cong \alpha_B \\
GFA & \xrightarrow{Gh} GFB
\end{align*}
\]

Then also $f = \alpha_B^{-1} GFF \alpha_A$.

So $GFF \circ Gf = Gh$.

As $G$ is faithful, $Ff = h$.

$F$ essentially surjective.

Given $B \in \text{ob } D$, we have an isomorphism $\beta_B : B \to FGB$.

(ii) Suppose $F$ is full, faithful, and essentially surjective. We construct a functor $G$ and a natural isomorphism $\beta : 1_D \to FG$.

For each $C \in \text{ob } D$ choose a pair $(GC, \beta_C)$.
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such that $\beta_c$ is an iso. $C \to FGC$ in $D$.

Given $h: C \to D$ in $D$, the composite

$$
\begin{array}{c}
\beta_c & \Rightarrow & \beta_d \\
\downarrow & & \downarrow \\
FGC & \Rightarrow & FGD
\end{array}
$$

$\beta_c$ is unique (or a unique $h: GC \to GD$ as $F$ is full and faithful).

We check whether $G$ really is a functor:

Given $h': D \to E$, both $G(h'h)$ and $Gh'Gh$ are the unique $f'$ that makes the diagram commute:

$$
\begin{array}{c}
\beta_c & \Rightarrow & \beta_d \\
\downarrow & & \downarrow \\
FGC & \Rightarrow & FGE
\end{array}
$$

So they must be equal.

By construction, $\beta$ is a nat. transf. $1_D \Rightarrow FG$. We obtain $\alpha_A$ from $\beta_{FA}: FA \to FGFA$ as $F$ is full and faithful. $\beta_{FA} = \#(\alpha_A)$ for a unique $\alpha_A: A \to GFA$.

The fact that $\alpha_A$ is an isomorphism and $\kappa$ is nat. follow from $F$ being full and faithful (exercise) $\square$

Examples

(a) The cat. $\text{Set}_B$ is equivalent to $\text{Set}^B$ (B-indexed families of sets). In one direction, the equivalence sends $(f_x)_{x \in A}$ to $(f(x))_{x \in A}$ and a morphism

$$
A \xrightarrow{h} A' \text{ to the family } (h(f(b)))_{b \in B}
$$
In the other direction, we send \((A_b | b \in B)\) to the disjoint union \( \coprod_{b \in B} A_b = U \{ A_b \times \{b\} \}\)
equipped with its projection to \(B\).

(b) For a field \(K\), the cats \(K\-\text{Mod}_{fd}\) and 
\((K\-\text{Mod}_{fd})^{op}\) (of finite dim vec. spaces and its opposite)
are equivalent. The functors in both directions are \(V \rightarrow V^*\), and the isomorphism \(V \rightarrow V^{**}\) is that of example (a).

(c) The cat \(\text{Mat}_K\) (from "unusual maps") is equivalent to \(K\-\text{Mod}_{fd}\). The functor 
\(F : \text{Mat}_K \rightarrow K\-\text{Mod}_{fd}\) sends \(N\) to \(K^N\) and a matrix \(M\) to the linear map it represents w.r.t. the standard bases. To define a functor \(G\) in the other direction, we need to choose a basis for each fin. dim. vector space \(GV = \text{dim} V\), 
\(G(f : V \rightarrow W)\) is the matrix representing \(f\) w.r.t. a choice of bases.

\(GF\) is the identity functor (if we choose the standard basis), and the chosen bases give us a nat. iso \(1 \rightarrow FG\)
E. Representable Functors

Recall the hom-functors \( C(A, B) : C \to \text{Set} \). We can put these together into a functor.

**Def.** Let \( C \) be a locally small cat. We define a functor
\[
Y : C^{\text{op}} \to [C, \text{Set}]
\]
called the Yoneda embedding, by setting
\[
YA = C(A, -) \quad \text{and} \quad Y(f : A \to B) \text{ is the natural transformation with components }
\]
\[
(Yf)_c : C(B, C) \to C(A, C)
\]

**Remark.** We could also define a similar functor
\[
C \to [C^{\text{op}}, \text{Set}]
\]
We should check that \( Yf \) really is a nat. transf., and \( Y \) is a functor.

Give \( f : A \to B, \ g : C \to D \), we need:
\[
\begin{align*}
C(B, C) & \xrightarrow{\sim} C(A, C) \\
C(B, D) & \xrightarrow{\sim} C(A, D)
\end{align*}
\]
\[
\xymatrix{ C(B, C) \ar[r]^{\sim} & C(A, C) \\
C(B, D) \ar[r]^{\sim} & C(A, D) } \]
\[
h : B \to C \text{ is sent to } g(hf) \text{ and } (gh)f
\]
respectively.

So by associativity of composition, this is a nat. transf. Similarly, assoc. of comp. also implies that \( Y \) is a functor.
What is so special about the hom-functors \( \mathcal{C}(A, -) \)? Given a natural transformation \( \alpha: \mathcal{C}(A, -) \to F \), let us look at the naturality square.

Let's "chase" the identity:

\[
\begin{array}{ccc}
\mathcal{C}(A, A) & \xrightarrow{\alpha_A} & FA \\
\downarrow_{1_A} & & \downarrow_{Ff} \\
\mathcal{C}(A, B) & \xrightarrow{\alpha_B} & FB \\
\end{array}
\]

for some \( f: A \to B \) in \( \mathcal{C} \)

We see that \( \alpha_B(f \cdot 1_A) = Ff(\alpha_A(1_A)) \) i.e. \( \alpha_B(f) \) is completely determined by \( \alpha_A(1_A) \), so \( \alpha \) itself is completely determined by \( \alpha_A(1_A) \in FA \).

**Theorem (Yoneda Lemma)**: Let \( \mathcal{C} \) be a locally small category, \( A \in \text{ob } \mathcal{C} \), and \( F: \mathcal{C} \to \text{Set} \) a functor. Then there is a bijection

\[
\Theta: \text{Nat}(\mathcal{C}(A, -), F) \to FA
\]

between the natural transformations \( \mathcal{C}(A, -) \to F \) and elements of \( FA \). Moreover, this bijection is natural in \( A \) and \( F \).

**Proof**: Given \( \alpha: \mathcal{C}(A, -) \to F \), we set \( \Theta(\alpha) = \alpha_A(1_A) \).

Given \( x \in FA \), we define a natural transformation \( \psi(x) \) by \( \psi(x)_B(f) = Ff(x) \) i.e. \( \psi(x)_B: \mathcal{C}(A, B) \to FB \)

\[
f \mapsto Ff(x)
\]
We check that $\Psi(x)$ is a nat. transf. Given $g : B \to C$, we have:

\[
\begin{array}{c}
\mathcal{C}(A, B) \xrightarrow{\psi(x)_B} FB \\
\mathcal{C}(A, C) \xrightarrow{\Psi(x)_C} FC \\
g \circ - \xrightarrow{g \circ f} Fg \\
F(g \circ f) = Fg \circ Ff \\
F(g \circ f)(x) = Fg \circ Ff(x) \\
\end{array}
\]

So $F(g \circ f)(x) = Fg \circ Ff(x)$ as $F$ is a functor so the square commutes.

We check that $\Theta$ and $\Psi$ are mutually inverse.

1. $\Psi(\Theta(x))_B = \Psi(\alpha_B(1_B))_B : \mathcal{C}(A, B) \to FB$

   sends $f$ to $Ff(\alpha_B(1_B)) = \alpha_B(f)$, so $\Psi(\Theta(x)) = \alpha$

2. $\Theta(\Psi(x)) = \Psi(\alpha_A(1_A)) = F\alpha_A(x) = 1_{FA}(x)$

We now fix $F$ and show $\Theta$ is a nat. in $A$.

Given $f : A \to B$, we have a square:

\[
\begin{array}{c}
\text{Nat}(\mathcal{C}(A, -), F) \xrightarrow{\Theta_A} FA \\
\text{Yoneda embedding} \downarrow \alpha \downarrow \alpha_B(1_a) \downarrow \mathcal{C}(B, -) \xrightarrow{\Theta_B} FB \\
\text{Nat}(\mathcal{C}(B, -), F) \xrightarrow{\Theta_B} FB \\
\end{array}
\]

\[
\begin{array}{c}
\text{Nat}(\mathcal{C}(A, -), F) \xrightarrow{\Theta_A} FA \\
\text{Nat}(\mathcal{C}(B, -), F) \xrightarrow{\Theta_B} FB \\
\end{array}
\]
Here \( \Theta_B(\alpha \circ Yf) = \alpha_B \circ (Yf)_B(1_B) \)

We have \( (Yf)_B = - \circ f \), so
\[
\alpha_B \circ (Yf)_B(1_B) = \alpha_B(f)
\]

On the other hand, \( Ff \circ \Theta_A(\alpha) = Ff \circ \alpha_A(1_A) = \alpha_B(f) \)
so the square commutes and \( \Theta \) is natural in \( A \).

Exercise: Show that \( \Theta \) is nat. in \( F \) for fixed \( A \).

\[ \square \]

Definition: A functor \( F : C \to \text{Set} \) is called \( \underline{\text{representable}} \) if it is isomorphic to \( C(A,-) \) for some \( A \in \text{ob} C \). A representation of \( F \) is a pair \( (A, \alpha) \) where \( A \in \text{ob} C \), \( \alpha \in FA \), and \( \psi(x) \) is a nat. iso. \( C(A,-) \to F \).

We also call \( A \) a universal element of \( F \).

Corollary: The Yoneda embedding is full & faithful.

Proof: Putting \( F = C(B,-) \) in the Yoneda lemma gives us a bijection between morphisms \( C(A,-) \to C(B,-) \) in \([C, \text{Set}^+]\) and elements of \( C(B,A) \), i.e. morphisms \( B \to A \) in \( C \).

The inverse is exactly the action of the Yoneda embedding on morphisms (check). This shows that the Yoneda embedding is full & faithful.
Corollary: ("Representations are unique up to unique isomorphism")

If \((A, x)\) and \((B, y)\) are both reps of \(F: C \to \text{Set}\), then there is a unique iso. \(f: A \to B\) in \(C\) with \(Ff(x) = y\).

Proof: We have a composite iso. \(C(B, -) \Rightarrow F \Rightarrow C(A, -)\).

As the Yoneda embedding is full & faithful, this is of the form \(Y(f)\) for a unique isomorphism \(f: A \to B\) in \(C\). (cf. Ex. Sh 1 Q 1\(\mathbb{C}\))

So \(Y(f) = \Psi(x)^{-1} \Psi(y)\), or \(\Psi(x) Yf = \Psi(y)\).

Via the bijection in the Yoneda lemma this is equivalent to \(Ff(x) = y\).

Examples: (a) The forgetful functor \(Gp \to \text{Set}\) is representable by \((\mathbb{Z}, 1)\), since homomorphisms \(f: \mathbb{Z} \to G\) correspond bijectively to elements \(f(1)\) of the underlying set of \(G\). Similarly, \(\text{Rng} \to \text{Set}\) is representable by \((\mathbb{Z}[x], x)\).

(b) The covariant powerset functor \(P: \text{Set} \to \text{Set}\) isn't representable (proof as exercise).

But \(D^+: \text{Set}^\text{op} \to \text{Set}\) is represented by \((\mathbb{Z}, \{1\})\) where \(2 = \{0, 1\}\) since subsets \(A' \subseteq A\) correspond bijectively to functions \(\chi_A: A \to 2\).
(c) The dual space functor \((-)^*: \text{K-Mod}^{\text{op}} \to \text{K-Mod}\), when composed with the forgetful functor \(\text{K-Mod} \to \text{Set}\), is representable by \((K, 1_k)\)

2. LIMITS AND COLIMITS

A. Terminal objects & Products

**Def:** A terminal object in a cat \(\mathcal{C}\) is an object \(1\) such that for every object \(A \in \mathcal{C}\) there is a unique morphism \(A \to 1\)

**Proposition** Any terminal object is unique up to unique isomorphism

**Proof** Suppose \(1\) and \(1'\) are terminal objects in \(\mathcal{C}\). Then there is a unique morphism \(f: 1 \to 1'\) and a unique \(g: 1' \to 1\). This gives a morphism \(gf: 1 \to 1\), but there is a unique morphism \(1 \to 1\) we must have \(gf = \text{id}_1\). Similarly \(fg = \text{id}_{1'}\).
So \(1\) and \(1'\) are isomorphic.

The dual notion is an initial object. \(0\) is initial if there is a unique morphism \(0 \to A\) for any \(A \in \mathcal{C}\).

**Examples:** In \(\text{Set}\), any 1-element set is terminal, and the empty set is initial.

In \(\text{Top}\), the 1-element top. space is terminal, and the empty one is initial.
Category Theory

In $\text{Gr}$, the one-element group is both initial and terminal. We write this as $0$ and call it a zero-object.

Similarly, $\text{K-Mod}$.

In $\text{Rng}$, the 1-element ring is terminal and $\mathbb{Z}$ is initial.

**Def:** A product of two-objects $A, B \in \text{obC}$ is a triple $(P, \pi_A, \pi_B)$ of an object $P \in \text{obC}$ and two morphisms $\pi_A : P \rightarrow A$ and $\pi_B : P \rightarrow B$ such that, if there is any other triple $(C, f : C \rightarrow A, g : C \rightarrow B)$ then there is unique morphism $c : C \rightarrow P$ such that $\pi_A c = f$ and $\pi_B c = g$.

![Diagram](image.png)

**Proposition:** A product of $A$ and $B$ is unique up to unique isomorphism.

**Proof:** Similar to terminal object, or note: A product of $A$ and $B$ is a rep. of the functor $C \mapsto C(C,A) \times C(C,B) : \text{obC} \rightarrow \text{Set}$.

We already saw that representations are unique up to unique isomorphism.

We write $A \times B$ for "the" product of $A$ and $B".$
Examples. In Set, the product of 2 sets $A \times B$ is their cartesian product.

In $Gp$, $R$-Mod, $Rng$, $Top$, ..., we can equip the cartesian product with the appropriate structure. In proofs, "and" is the product.

This generalises to products of any family of objects. The dual notion is a coproduct:

$$(A \times B, c_A, c_B) \text{ with } c_A : A \to A \times B \text{ and } c_B : B \to A \times B \text{ such that for any } C \text{ with } f : A \to C \text{ and } g : B \to C \text{ there is a unique } h : A \times B \to C \text{ s.t. } h c_A = f, h c_B = g$$

Examples. In Set, the coproduct $A \times B$ is the disjoint union $A \sqcup B : A \sqcup B$. The same is true for $Top$, but not in $Gp$:

there the coproduct is the free product $A * B$.

In $R$-Mod (and Ab$Gp$) the coproduct is the same as the product. We also call it a biproduct or direct sum and write $A \oplus B$.

In proofs the coproduct is "or".
Category Theory

B. Cones and Limits

**Def** Let \( J \) be a particular cat (usually small, often finite).

A diagram of shape \( J \) in \( \mathcal{C} \) is a functor \( J \to \mathcal{C} \).

Remember the examples of finite cats from §14.

If \( J = ( \bullet \to \bullet ) \), a diagram of shape \( J \) is a pair of parallel arrows \( A \xrightarrow{f} B \) in \( \mathcal{C} \).

If \( J = ( \bullet \rightrightarrows \bullet ) \), then a diagram of shape \( J \) is a comm. square \( A \xrightarrow{f} B \xleftarrow{g} C \xrightarrow{h} D \) in \( \mathcal{C} \).

**Def** Let \( D : J \to \mathcal{C} \) be a diagram.

A cone over \( D \) is an object \( A \in \text{ob}\mathcal{C} \) together with morphisms (called legs) \( \mu_j : A \to D(j) \) for all \( j \in \text{ob}\mathcal{J} \) s.t.

for any morphisms \( \kappa : j \to j' \) in \( J \), the triangle

commutes

i.e. \( D(A) \cdot \mu_j = \mu_{j'} \)
Remark A cone is really a special sort of nat. transf. Consider the const. functor \( \Delta A : \mathcal{J} \to \mathcal{C} \) which sends each \( j \in \text{obj} \mathcal{J} \), and each morphism \( \alpha \to 1 \) in \( \mathcal{C} \). Then a cone is a nat. transf. \( \mu : \Delta A \to \mathcal{D} \)

Def Given two cones \( (A, \mu) \) and \( (B, \nu) \) over a diagram \( D \), a morphism of cones is a morphism \( f : A \to B \) such that

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{\mu_j} & & \downarrow{\nu_j} \\
D(j) & & \\
\end{array}
\]

commutes \( \forall j \in \text{obj} \mathcal{J} \).

The cones over a particular diagram form a category.

Def A limit of \( D \) is a terminal cone, i.e. a terminal object of this category of cones. (Often written \( \lambda j : L \to D(j) \) \( \forall j \in \text{obj} \mathcal{J} \))

\[
\begin{array}{ccc}
A & \longrightarrow & L \\
\end{array}
\]

Dually we have cocones under a diagram \( D \) (some people just say "cone under \( D \)") and a colimit is an initial cocone.

Proposition Limits and colimits are unique up to unique isomorphism.

Proof: exercise.
Category Theory

So we can speak of "the" limit of $D$ (if it exists).

We say $C$ has limits of shape $J$ if any diagram $D : J \to C$ has a limit.

Examples: (a) A terminal object is the limit of the empty diagram. A product is the limit of a discrete diagram with two objects. More generally, we say product to the limit of any discrete diagram.

We write $\prod_{j \in J} D(j)$ (or e.g. $\prod_{i \in M} A_i$)

The legs are called product projections.

(b) The limit of a diagram of shape $\begin{array}{cc} & & \vdots \\ & \mu_k \\ \mu_l & \downarrow & \\ A & \longrightarrow & B \end{array}$ is called an equaliser. Given a pair of arrows $A \xrightarrow{f} B$ in $C$, a cone over this diagram is $\begin{array}{cccc} & & & \mu_k \\ & & \mu_l \\ & \downarrow & \downarrow \\ A & \longrightarrow & B & \longrightarrow \\ \downarrow & \mu_j \\ C \end{array}$ such that $\mu_k = f \mu_j = g \mu_j$.

Or (simpler) just $\begin{array}{ccc} C & \xrightarrow{c} & A \xrightarrow{f} B \end{array}$ with $tc = gc$.

A limit cone is a universal pair $(E, e)$

$\begin{array}{ccc} E & \xrightarrow{e} & A \xrightarrow{f} & B \\ & \downarrow & \downarrow & \downarrow \\ & c & \downarrow & \mu_j \\ & & C & \\ \end{array}$

A colimit of this diagram is called a coequaliser. In $\text{Set}$, the equaliser of $f$ and $g$ is the set $E = \{ a \in A | f(a) = g(a) \}$.
equipped with the inclusion map into \( A \).

(c) The limit of a diagram of shape \( \bullet \to \bullet \) is called a pullback. A cone over such a diagram is just a commutative square

\[
\begin{array}{ccc}
\ast & \to & A \\
\downarrow_{\mu_1} & & \downarrow_{\mu_3} \\
B & \to & C
\end{array}
\]

\[ \mu_3 = f \mu_1 = g \mu_2 \]

i.e. the square commutes.

We write a pullback square as follows:

\[
\begin{array}{ccc}
P & \to & A \\
\downarrow_{f} & & \downarrow_{\pi_1} \\
B & \to & C
\end{array}
\]
or

\[
\begin{array}{ccc}
A \times_B B & \to & A \\
\downarrow_{\pi_2} & & \downarrow_{f} \\
B & \to & C
\end{array}
\]

Pullbacks are also called fibred products.

We say \((A \times_B B, \pi_1, \pi_2)\) is the pullback of \( f \) and \( g \) along \( f \). A pullback of \( f \) with itself is also called kernel pair of \( f \)

In \text{Set}, we can construct pullbacks by first forming the product \( A \times B \) and then the equaliser \( P \to A \times B \) of \( A \times B \overset{f}{\underset{g}{\rightharpoonup}} C \), i.e. the set \( \{ (a, b) \in A \times B \mid f(a) = g(b) \} \).
Category Theory

Recall: pullbacks are limits of a diagram of shape

Notice that a colimit under this diagram is trivial.

The appropriate dual is a pushout: the colimit of a diagram of shape

\[ \begin{array}{ccc}
\bullet & \longrightarrow & A \\
\downarrow & & \downarrow g \\
C & \rightarrow & D
\end{array} \]

**Theorem** "Constructing limits"

(i) If \( C \) has equalisers and all small products, then \( C \) has all small limits.

(ii) If \( C \) has equalisers and finite products, then \( C \) has all finite limits.

(iii) If \( C \) has pullbacks and a terminal object, then \( C \) has all finite limits.

**Proof:** (i) and (ii). Let \( D: J \rightarrow C \) be a diagram with \( J \) small (resp. finite). Form the products

\[ P = \prod_{j \in \mathrm{obj} J} D(j) \quad \text{and} \quad Q = \prod_{\alpha \in \mathrm{mor} J} D(\alpha \circ \alpha) \]

and the morphisms \( P \xrightarrow{f} Q \) defined by \( \pi_\alpha f = \pi_\alpha \circ \alpha \) and

\[ \pi_\alpha q = D(\alpha) \pi_{\alpha \circ \alpha} \quad \text{for} \quad \alpha \in \mathrm{dom} \alpha. \]

\[ \begin{array}{ccc}
P & \xrightarrow{f} & Q \\
\pi_{j'} & \downarrow & \pi_{\alpha} \\
D(j') & \xrightarrow{\pi_j} & D(\alpha)
\end{array} \]
let \((L,e)\) be an equaliser of the pair \((f,g)\)

\[
L \xrightarrow{e} P \xrightarrow{f} Q
\]

We claim that the family \(\{ \pi_j e = \lambda_i : L \to D(j) \mid j \in \text{ob} \mathcal{G} \}\)
forms a limit cone on \(D\).

It is indeed a cone, as \(D(\alpha) \lambda_j = D(\alpha) \pi_j e = \pi_x g e = \pi_x f e = \pi_j e = \lambda_j\).

Given a cone \(\{ \mu_j : M \to D(j) \mid j \in \text{ob} \mathcal{G} \}\) we have a unique morphism \(m : M \to P\) satisfying \(\pi_j m = \mu_j \forall j \in \text{ob} \mathcal{G}\).

Then \(f m = g m\), as \(\pi_a f m = \pi_x g m \forall a \in \text{mor} \mathcal{G}\).

(Exercise: check)

So we have a unique map \(n : M \to L\) satisfying \(\lambda_j n = \mu_j \forall j\).

(iii) It is enough to construct equalisers and finite products.

Any finite product \(\prod_{i} A_i\) can be constructed from products of pairs \(((A_i \times A_2) \times A_3) \times A_4 \ldots\)

The product of the empty family is the terminal object 1.

Given 2 objects \(A\) and \(B\), their product can be constructed as a pullback of \(B \rightarrow\)
Category Theory

Given a pair of parallel morphisms \( A \xrightarrow{f} B \), their equaliser can be constructed as the pullback \( K \) of

\[
\begin{array}{ccc}
A & \xrightarrow{g} & B \\
\downarrow{h} & & \downarrow{k} \\
A & \rightarrow & A \times B
\end{array}
\]

satisfying \( h=k \) and \( fh=gk \), so it's equiv. to a cone over \( A \xrightarrow{f} B \).

The categories \( \text{Set} \), \( \text{Grp} \), \( \text{Ring} \), \( \text{R-Mod} \), \( \text{Top} \) ... have all small products and equalisers, so they all have all small limits.

We call a category with all small limits complete, and a category with all finite limits \textit{finitely complete}.

Similarly, the categories have small coproducts and small coequalizers, so they are \textit{cocomplete}.

C. Special morphisms

\textbf{Definition}. A morphism \( f : A \rightarrow B \) in a cat \( \mathcal{C} \) is a \textit{monomorphism} if, given any \( C \xrightarrow{g} A \) with \( fg=fh \), we necessarily have \( g=h \) (\( f \) is left-cancelled).

Dually, \( f \) is an \textit{epimorphism} if given \( B \xrightarrow{k} D \) with \( kf=lf \), we necessarily have \( k=l \).

\textbf{Examples}. In \( \text{Set} \), monomorphisms are injective \( \text{fn}s \) and epimorphisms are surjective functions.
In $\text{Gp}$, monos are injective homomorphisms, epis are surjective homomorphisms. Similarly in $\text{Top}$. 

However, it's not always this simple.

For example $\mathbb{Z} \to \mathbb{Q}$ is an epi in $\text{CRng}$, and in $\text{Mon}$ the inclusion $(\mathbb{N},+) \to (\mathbb{Z},+)$ is epic.

**Proposition:** If $f : A \to B$ and $g : B \to A$ satisfy $gf = 1_A$, then $f$ is monic, and $g$ is epic.

**Proof:** If we have $C \xrightarrow{h} A$ with $fh = fK$, then also $gfh = gfK$, i.e., $h = K$ so $f$ is monic.

$g$ epic in $C \iff g$ monic in $C^{op}$

**Definition** (a) If $gf = 1_A$ as above, we call $f$ a split monomorphism and $g$ a split epimorphism.

(b) We say $f : A \to B$ is a regular monomorphism if it is an equaliser of some pair $B \xrightarrow{f} C$.

Dually, a regular epimorphism is the coequaliser of some pair $D \xleftarrow{g} A$.

**Exercise:** • Prove that any regular monomorphism is indeed monic.
• Prove that every split monomorphism is a regular monomorphism (consider $fg$ and $1_b$).
In $\textbf{Set}$, every mono is a regular mono, but not in $\textbf{Top}$. (In $\textbf{Top}$, regular monos are injections $f: Y \to X$ for which $Y$ has the subspace topology of $f(X)$.)

In $\textbf{Set}$, every mono with a non-empty domain is split, and the fact that every epi in $\textbf{Set}$ is split is equivalent to the axiom of choice.

In $\textbf{k-Mod}$, all monos and epis are split.

**Proposition** If $f$ is both an epi and regular monic, it is an iso.

**Proof.** If $f: A \to B$ is the equaliser of $B \xrightarrow{g} C$, then $g=h$ as $f$ is epic. But $1_A$ is an equaliser of $(g,g)$, so by uniqueness of limits, $f$ is an iso.

**Definition:** A category is called balanced if every morphism which is monic and epic is an iso.

$\textbf{Set}$ and $\textbf{Gp}$ are balanced, $\textbf{Mon}$ and $\textbf{Top}$ aren't. ($\textbf{Top}$: continuous bijections need not be homeomorphisms)

In diagrams, we write $A \xrightarrow{f} B$ for monos, and $A \xrightarrow{f} B$ for epis.
Lemma. "Pullbacks preserve monos"

Given a pullback square: \[
\begin{array}{ccc}
P & \rightarrow & A \\
\downarrow^h & & \downarrow^f \\
B & \rightarrow & C \\
\downarrow^g & & \downarrow^f \\
\end{array}
\]

if \( f \) is monic, then \( k \) is monic.

Proof: Suppose \( D \rightarrow P \) satisfy \( kl = km \).

\[
\begin{array}{ccc}
D & \rightarrow & P \\
\downarrow^l & & \downarrow^h \\
B & \rightarrow & C \\
\downarrow^g & & \downarrow^f \\
\end{array}
\]

Then \( fhl = gkl = gkm = fhm \).

So \( hl = hm \) (as \( f \) is monic)

So \( l \) and \( m \) correspond to the same cone over \( A \), and

hence \( l = m \). \( \Rightarrow \) \( k \) monic \( \square \)

Def. A subobject of an object \( A \) in a cat \( \mathcal{C} \) is either a monomorphism \( A' \rightarrow A \) in \( \mathcal{C} \), or an isomorphism class (in \( \mathcal{C}/A \)) of such monomorphisms. We write \( \text{Sub}_A \) for the full subcategory of \( \mathcal{C}/A \) whose objects are the monomorphisms \( A' \rightarrow A \).

Note that this is a preorder.

A cat \( \mathcal{C} \) is well-powered if each \( \text{Sub}_A \) is equivalent to a partially ordered set, i.e. \( \exists \) a set \( \{ A_i \rightarrow A \mid i \in I \} \) of monomorphisms meeting each isomorphism class in \( \text{Sub}_A \).

Example. Set is well-powered since \( \text{Sub}_A \) is PA.

Similarly, Grp, Rng, Top are all well-powered.
Category Theory.

D. Preserving Limits

**Def** Let $F : C \to D$ be a functor.

(a) We say $F$ preserves limits of shape $\mathcal{J}$ if, given any $D : \mathcal{J} \to C$ and a limit cone $(L \triangleright D(j), j \in \mathcal{J})$ for $D$, the cone $(F L \triangleright F D(j), j \in \mathcal{J})$ is a limit for $F D$.

(b) We say $F$ reflects limits of shape $\mathcal{J}$ if, given $D : \mathcal{J} \to C$ and a cone $(L \triangleright D(j), j \in \mathcal{J})$ such that $(F L \triangleright F D(j), j \in \mathcal{J})$ is a limit of $F D$ in $D$, then $(L, \lambda j)$ forms a limit for $D$.

(c) We say $F$ creates limits of shape $\mathcal{J}$ if, given $D : \mathcal{J} \to C$ and a limit $(M \mu j \to F D(j), j \in \mathcal{J})$ for $F D$, there exists a cone $(L \triangleright D(j), j \in \mathcal{J})$ over $D$ whose image is isomorphic to $(M, \mu j)$, and any such cone is a limit in $C$.

**Corollary** In any of the versions of the "constructing limits" theorem, we can replace "$C$ has" with either "$C$ has and $F : C \to D$ preserves" or "$D$ has and $F : C \to D$ creates".

**Proof**: Exercise.

**Examples"creating limits":**

(a) The forgetful functor $G \to \text{Set}$ creates all small limits. For example, if $\{G_j \mid j \in \mathcal{J}\}$ is a family of groups, then the product set $\Pi G_j$ has a unique group structure making
The projections into homomorphism and this structure makes it into a product in \( \text{Gp} \). But \( \text{Gp} \rightarrow \text{Set} \) doesn't preserve coproducts (or other colimits).

(b) The forgetful functor \( \text{Top} \rightarrow \text{Set} \) preserves all small limits and colimits, but it doesn't reflect them: given spaces \( X \) and \( Y \), there are (in general) other topologies on the set \( X \times Y \), making the projections continuous but not making it into a product in \( \text{Top} \).

(c) The inclusion functor \( \text{AbGp} \rightarrow \text{Gp} \) reflects coproducts but doesn't preserve them. A coproduct \( \bigoplus_{i \in I} A_i \) in \( \text{Gp} \) is non-abelian unless all but one of the \( A_i \) are trivial, and then it coincides with the coproduct in \( \text{AbGp} \).

(d) Let \( C \) be a cat and \( B \in \text{Ob} C \). The forgetful functor \( C/B \rightarrow C \) sending \( \frac{A}{B} \) to \( A \) creates all colimits which exist in \( C \). A diagram \( D : G \rightarrow C/B \) is essentially a diagram \( UD \) of shape \( G \) in \( C \) together with a cocone \( (UD(i) \rightarrow B)_{i \in G} \).

Given a colimit cocone \( (UD(i) \rightarrow L) \) for \( UD \), we get a unique \( L \rightarrow B \) making all the \( UD(i) \rightarrow L \) into morphisms in \( C/B \), which "lifts" the colimit cocone to a colimit cocone in \( C/B \).

However, \( C/B \rightarrow B \) doesn't preserve all limits.
Category theory.

E.g. if $A \cong B$ and $g : B \to B$ are objects of $S / B$, their product in $S / B$ is the diagonal of the pullback square

$$
p \leftarrow A
\downarrow \downarrow \downarrow
\downarrow
\downarrow
\downarrow
\downarrow
$$

(If the pullback exists in $S$, and $P \neq C \times A$ in general)

* (e) "Limits in functor categories are constructed object by object."

Let $C$ and $D$ be categories. Write $\text{ob}^D$ for the category of functors from the discrete category on the objects of $D$ to $C$, or the product of $D$ copies of $C$.

Then the forgetful functor $U : [D, C] \to \text{ob}^D$ creates all limits (and colimits) that exist in $C$.

To see this, let $D : J \to [D, C]$ be a diagram in the functor category, and suppose that for every object $A$ in $D$ the diagram $UD_A$ (UD evaluated at $A$) has a limit in $C$, $(LA, \lambda_A)$.

Then clearly $L : \text{ob}^D \to C$ is a limit of $UD$. 
We want to show that $L$ is actually a functor $L : \mathcal{C} \to \mathcal{D}$ and is the limit of $D$ in $[\mathcal{D}, \mathcal{C}]$.

Given a morphism $f : A \to B$ in $\mathcal{D}$, we have, for any morphism $j \in \mathcal{J}$ a commutative square

$$
\begin{align*}
D(j)A & \xrightarrow{D(\alpha)A} D(j')A \\
\downarrow D(j)f & \quad \downarrow D(j')f \\
D(j)B & \to D(j')B
\end{align*}
$$

So $(L_A, D(j)f \cdot \lambda j^A)$ forms a cone on $UD_B^C$, which gives a unique morphism $Lf : L_A \to L_B$. 

Category Theory

making \[ L A \xrightarrow{L f} L B \]

\[ \lambda_j^A \downarrow \quad \lambda_j^B \]

\[ D(j)A \rightarrow D(j)B \]

This makes \( L \) into a functor \( \mathcal{D} \rightarrow \mathcal{C} \), the \( \lambda_j \) into nat. transf. \( L \rightarrow D(j) \) and \( L \) into the limit of \( D \) in \( [\mathcal{D}, \mathcal{C}] \) (check all this).

Note that this also shows that the functor "evaluation at \( A \)" \( \text{ev}_A : [\mathcal{D}, \mathcal{C}] \rightarrow \mathcal{C} \) preserves limits.

Remark "Monos in functor categories".

In any cat, a morphism \( f : A \rightarrow B \) is monic iff \( 1_A \xrightarrow{1_A} f \) is a pullback

\[ \begin{array}{c}
  A \\
  \downarrow f \\
  B
  \end{array} \]

(i.e. iff its kernel pair is \( (A, 1_A, 1_B) \)).

Hence a functor which preserves pullbacks must preserve monos. Therefore a morphism \( \alpha : F \rightarrow G \) in a functor cat \([\mathcal{D}, \mathcal{C}]\) is monic \( \Leftrightarrow \) each component \( \alpha_c : FC \rightarrow GC \) is a mono in \( \mathcal{C} \). (cf Ensh 1Q7)

There is a connection between initial objects and limits.

Lemma ("initial object as limit") Let \( \mathcal{C} \) be an arbitrary cat. Then \( \mathcal{C} \) has an initial object iff the diagram \( 1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C} \) has a limit.

Proof \( \Rightarrow \) Let \( I \) be the initial object. Write \( \lambda_A : I \rightarrow A \) for the unique morphism from \( I \) to each object \( A \). Then we claim that \( (I, \lambda_A) \) forms a terminal cone.
on $1_{\mathcal{E}}$. Indeed it is a cone, as \( \lambda \) commutes for each $f$ in $\mathcal{E}$, by uniqueness of $\lambda_B$.

Given another cone $(B, \mu_A)$ over $1_{\mathcal{E}}$, the morphism $\mu_I : B \to I$ satisfies $B \xrightarrow{\mu_I} I$ for all $A$, so $\mu_I$ is a morphism of cones. But any morphism of cones $\nu$ satisfies $B \xrightarrow{\nu} I$.

So $\nu = \mu_I$, so $\mu_I$ is the unique morphism of cones. $\Rightarrow$ $(I, \lambda_A)$ is a limit as claimed.

• " \( \implies \) " If we have a limit $(I, \lambda_A)$ for $1_{\mathcal{E}}$, we want to show $I$ is initial. As we already have morphisms $\lambda_A : I \to A$, we must show that these are unique, i.e., given $f : I \to A$, we have $f = \lambda_A$ we certainly have $\lambda_A : I \to A$

$f \lambda_I = \lambda_A$, so it's enough to show $\lambda_I = 1_I$

Putting $f = \lambda_A$ we get $\lambda_A \lambda_I = \lambda_A$ for all objects $A$, so $\lambda_I$ is a morphism of cones from the limit cone to itself, so we have $\lambda_I = 1_I$ \( \square \)
Category Theory

E. Projectives

**Def** An object $P$ of a cat $\mathcal{E}$ is projective if given any diagram $\begin{array}{c} A \\ \downarrow h \\ B \end{array} \xrightarrow{f} \begin{array}{c} A \\ \downarrow g \\ P \end{array}$ with $f$ epic, there exists $h: P \to A$ with $fh = g$.

Dually $I$ is injective in $\mathcal{E}$ if it is projective in $\mathcal{E}^{op}$.

**Remark** Note that $h$ need not be unique. If $\mathcal{E}$ is locally small, $P$ is projective iff $\mathcal{E}(P, -)$ preserves epics.

**Lemma** For any locally small $\mathcal{E}$, all representable functors are projective in $[\mathcal{E}, \text{Set}]$.

**Proof**: The dual of "monos in functor cats" says that $\alpha: F \to G$ is epic in $[\mathcal{E}, \text{Set}]$ iff $\alpha^*: FA \to GA$ is surjective for all $A$.

Now, given $\begin{array}{c} F \\ \alpha \\ \downarrow \beta \\ G \end{array}$ by the Yoneda lemma, $\beta$ corresponds to $y \in GA$.

As $\alpha$ is epic, $\exists x \in FA$ with $\alpha_A(x) = y$.

Then $x$ corresponds to $\begin{array}{c} F \\ \xrightarrow{\alpha} \\ \text{subjectively} \end{array}$

$\square$
Lemma A coproduct of projectives is projective.

Proof: exercise.

Examples: In Set, every object is projective (as any epi is split, which uses the axiom of choice).

In $Gp$, any free group is projective, and in fact these are the only projective objects in $Gp$.

In $R$-Mod a module is projective iff it is a direct summand of a free module.

3. ADJUNCTIONS

A. DEFINITIONS & EXAMPLES

Def (D.M. Kan) Let $F : C \to D$ and $G : D \to C$ be two functors. An adjunction between $F$ and $G$ is a specification, for each pair of objects $(A \in ob C, B \in ob D)$, of a bijection between morphisms $FA \to B$ and morphisms $A \to GB$ in $C$, which is natural in $A$ and $B$.

If $C$ and $D$ are locally small, this means that the functors $D^{op} \times D \to Set$ sending $(A, B)$ to $D(FA, B)$ and to $C(A, GB)$ are naturally isomorphic.

We say $F$ is left adjoint to $G$, or $G$ is right adjoint to $F$, and write $F \dashv G$. 
Notation: Given \( \frac{F}{G} \), we sometimes write:

\[
\begin{align*}
FA & \rightarrow B \\
A & \rightarrow GB
\end{align*}
\]

for the bijection,

and we write \( \tilde{f} : A \rightarrow GB \) for the morphism corresponding to \( f : FA \rightarrow B \) and \( \overline{g} : FA \rightarrow B \) corresponds to \( g : A \rightarrow GB \). Note \( \tilde{f} = f \).

Examples:
(a) The free functor \( F : \text{Set} \rightarrow \text{Gp} \) is left adjoint to the forgetful functor \( G : \text{Gp} \rightarrow \text{Set} \) as homomorphisms \( FA \rightarrow B \) are uniquely determined by mappings \( A \rightarrow GB \). Similarly, for free rings, free \( R \)-modules, etc.
(b) The forgetful functor \( \text{Top} \rightarrow \text{Set} \) has both left and right adjoints: left \((0)\) equips a set \( A \) with its discrete topology since all functions \( DA \rightarrow X \) \((\text{for } X \text{ arbitrary space})\) are continuous; right \((I)\) equips \( A \) with the indiscrete topology.
(c) The functor \( \text{ob} : \text{Cat} \rightarrow \text{Set} \) has a left adjoint \( D \) sending a set \( A \) to the discrete category \( DA \), since a functor \( DA \rightarrow B \) is determined by its effect on objects. "ob" also has a right adj. \( I \), which sends \( A \) to the cat with objects given by the elements of \( A \), and exactly one morphism \( a \rightarrow b \) for each pair \( (a,b) \in A \times A \). \( D \) itself also has a left adj. \( \Pi_0 \). \( \Pi_0(B) \) is the set of connected components of \( B \), i.e. the quotient of \( \text{ob} B \) by the smallest equiv. relation which identifies \( c \) and \( d \) whenever \( \exists \) a morphism \( c \rightarrow d \) in \( B \). (given \( F : \mathcal{C} \rightarrow \mathcal{D} \), \( F \) is necessarily const. on each connected component of \( \mathcal{C} \) as each morphism must go to an identity morphism. So \( F \) induces a function \( \Pi_0 \mathcal{C} \rightarrow \mathcal{A} \).)
(d) Let \( 1 \) denote the category with one object \( * \) and one morphism. A functor \( F: 1 \to \mathcal{C} \) picks out one object in \( \mathcal{C} \).

Note: Only one functor from \( \mathcal{C} \) to \( 1 \).

This \( F \) is left adjoint to the unique functor \( \mathcal{C} \to 1 \) \( \iff \) \( F^* \) is an initial object of \( \mathcal{C} \).

\( F \) is right adjoint to \( \mathcal{C} \to 1 \) \( \iff \) \( F^* \) is a terminal object of \( \mathcal{C} \).

(e) Let \( \text{idem} \) be the cat with objects being pairs \( (A, e) \) where \( A \) is a set and \( e: A \to A \) satisfies \( e^2 = e \) (i.e., it's idempotent).

Morphisms \( (A, e) \to (A', e') \) are functions \( f: A \to A' \) satisfying

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A' \\
\downarrow e & & \downarrow e' \\
A & \xrightarrow{f} & A'
\end{array}
\]

We have a functor \( F: \text{Set} \to \text{idem} \) sending \( A \) to \( (A, 1_A) \), and a functor \( G: \text{idem} \to \text{Set} \) sending \( (A, e) \) to \( \{ e(a) \mid a \in A \} = \{ a \in A \mid e(a) = a \} \) (the image of \( e \), or the fixed pts of \( e \)).

\( G \) is both left and right adjoint to \( F \).

- morphisms \( f: (A, 1_A) \to (B, e) \) must satisfy

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow 1_A & & \downarrow e \\
A & \xrightarrow{f} & B
\end{array}
\]

This gives a bijection \( (A, 1_A) \to (B, e) \)

\[
A \mapsto \{ e(b) \mid b \in B \}
\]

- morphisms \( f: (B, e) \to (A, 1_A) \) must satisfy
(b) Let $X$ be a topological space, $\mathcal{P}X$ be the ordered set of closed subsets of $X$ and $\mathcal{P}X$ the set of all subsets of $X$.

The inclusion $\mathcal{P}X \hookrightarrow \mathcal{P}X$ has left adj. $A \mapsto \overline{A}$.

Since $\forall$ closed set $F$ we have $A \subseteq F \iff \overline{A} \subseteq F$.

(An adj. btw posets $P \sim Q$ always looks like:

$Fa \leq b \iff a \in Gb$)

(9) ("Adj. of contravariant functors")

Consider two sets $A, B$ and a relation $R \subseteq A \times B$.

We have a mapping $r: PA \rightarrow PB$ sending $A' \mapsto r(A') = \{ b \in B \mid \forall a \in A', (a, b) \in R \}$.

$l: PB \rightarrow PA$ sending $B' \mapsto l(B') = \{ a \in A \mid \forall b \in B', (a, b) \in R \}$.

$r$ and $l$ are contravariant functors between posets, and we have:

$A' \subseteq l(B') \iff A' \times B' \subseteq R \iff B' \subseteq r(A')$.

We can regard $l: PB \rightleftarrows PA^\circ$ as left adj. to $r: PA^\circ \rightleftarrows PB$.

(We sometimes say that $l$ and $r$ are contravariant functors adjoint on the right)

(h) The contravariant powerset functor $P^\circ: \text{Set}^\circ \rightarrow \text{Set}$ is right adjoint to $P^\circ: \text{Set} \rightarrow \text{Set}^\circ$.

Since functions $A \rightarrow PB$ correspond to relations $R \subseteq A \times B$, and hence to functions $B \rightarrow PA$. 

B. Properties

What does the naturality in $A$ and $B$ of the bijection $\frac{F_A \Rightarrow B}{A \Rightarrow GB}$ mean? We should have some kind of commutative square.

Naturality in $A$ says that for $a: A \rightarrow A'$ in $G$:

\[
\begin{array}{ccc}
\mathcal{E}(A, GB) & \overset{\pi}{\longrightarrow} & \mathcal{D}(FA, B) \\
\downarrow \circ a & & \downarrow \circ Fa \\
\mathcal{E}(A', GB) & \overset{u}{\longrightarrow} & \mathcal{D}(FA', B)
\end{array}
\]

and naturality in $B$ says that for $b: B \rightarrow B'$ in $D$

\[
\begin{array}{ccc}
\mathcal{D}(FA, B) & \overset{\pi}{\longrightarrow} & \mathcal{E}(A, GB) \\
\downarrow b \circ f & & \downarrow Gb \circ f \\
\mathcal{D}(FA', B') & \overset{\pi}{\longrightarrow} & \mathcal{E}(A, GB')
\end{array}
\]

So $g \circ a = g \circ Fa$ and $b \circ f = Gb \circ f$.

So in fact we have natural transformations like the ones appearing in the Yoneda lemma.

\[
\mathcal{D}(FA, -) \rightarrow \mathcal{E}(A, GB)
\]

\[
\mathcal{E}(-, GB) \rightarrow \mathcal{D}(F, -)
\]

So these isomorphisms are completely determined by where the identity goes.
Category Theory

$\eta_A : A \rightarrow GFA$ corresponds to $\eta_A : A \rightarrow GFA$

Any $F(A) \rightarrow B$ corresponds to $A \xrightarrow{\eta_A} GFA \xrightarrow{Gf} GB$

$1_{GB} : GB \rightarrow GB$ corresponds to $\epsilon_B : FGB \rightarrow B$

Any $g : BA \rightarrow GB$ corresponds to $FA \xrightarrow{\epsilon_B} FGB \xrightarrow{e_B} B$

Remember: $g \circ a = \overline{g} \circ F_a$

$b \circ f = Gb \circ f$

$FA \xrightarrow{f} B \leftrightarrow A \xrightarrow{\eta_A} GFA$

$\downarrow Gf$

$\overline{GB}$

$A \xrightarrow{g} GB \leftrightarrow FA \xrightarrow{Fg} FGB$

$\downarrow \epsilon_B$

$B$

Lemma The $\eta_A : A \rightarrow GFA$ form a natural transformation $\eta : 1_\text{FA} \rightarrow GF$. (Dually, the $\epsilon_B$ form a nat. trans. $\varepsilon : FG \rightarrow 1_\text{FB}$)

Proof: Given $a : A \rightarrow A'$, we have

$A \xrightarrow{\eta_A} GFA \xrightarrow{F\eta_A} GFA'$ corresponding to $FA \xrightarrow{F\eta_A} FA'$

$A \xrightarrow{a} A' \xrightarrow{\eta_{A'}} GFA' \leftrightarrow FA \xrightarrow{F\eta_A} FA' \xrightarrow{F\eta_{A'}} FA'$

So $A \xrightarrow{a} A'$ commutes, and $\eta$ is natural. □
Notation: Given a functor \( G : \mathcal{D} \to \mathcal{C} \) and an object \( A \) of \( \mathcal{C} \), we write \((A \downarrow G)\) for the category whose objects are pairs \((B, f)\) with \( B \in \text{ob} \mathcal{D} \) and \( f : A \to GB \) in \( \mathcal{C} \), and whose morphisms \((B, f) \to (B', f')\) are morphisms \( g : B \to B' \) in \( \mathcal{D} \) s.t.

\[
\begin{array}{c}
A \xrightarrow{f} GB \\
\downarrow \quad \downarrow \\
G B \xrightarrow{Gg} GB'
\end{array}
\]

commutes (Similarly, there's a category \((G \downarrow A)\)).

Thm "Adjunctions via initial objects"

Let \( G : \mathcal{D} \to \mathcal{C} \) be a functor. Then specifying a left adjoint for \( G \) is equivalent to specifying, for each \( A \in \text{ob} \mathcal{C} \), an initial object of \((A \downarrow G)\).

Proof "⇒" let \( F : \mathcal{C} \to \mathcal{D} \) be the left adjoint of \( G \). We show that \((FA, \eta_A)\) is initial in \((A \downarrow G)\).

Given one object \((B, f)\) in \((A \downarrow G)\), the triangle

\[
A \xrightarrow{f} GB
\]

\[
\eta_A \downarrow \quad \downarrow \quad \downarrow \\
GFA \xrightarrow{Gf} GB
\]

commutes iff

\[
\begin{array}{c}
FA \\
\downarrow \quad \downarrow \\
GFA \xrightarrow{Gh} GB
\end{array}
\]

commutes.

\[
\begin{array}{c}
FA \xrightarrow{f} B
\end{array}
\]

So \( F! \) morphism \( h : (FA, \eta_A) \to (B, f) \) in \((A \downarrow G)\), namely \( f \). □
"\(\Leftarrow\) Given an initial object \((F,A,\eta_A)\) in each \((A \downarrow G)\), we already know the action of \(F\) on objects. We want to see what \(F\) does on morphisms, that it is a functor and that it is adjoint to \(G\).

Given \(f : A \to A'\), we have an object \((A \xrightarrow{f} A' \quad \eta_A \eta_{A'}^{-1} GFA')\) in \((A \downarrow G)\), so \(\exists! \ g : FA \to FA'\) making

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A' \\
\downarrow \eta_A & & \downarrow \eta_{A'} \\
GFA & \xrightarrow{Gg} & GFA'
\end{array}
\]

commute. We set \(Ff = g\).

Uniqueness of \(g\) makes \(F\) functorial (check!)

To see that \(F\) is adjoint to \(G\), take any \(h : FA \to B\). Then the composite \(A \xrightarrow{\eta_A} GFA \xrightarrow{Gh} GB\) is a morphism \(A \to GB\).

Given \(K : A \to GB\), \(\exists! \ h : FA \to B\) making

\[
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & GB \\
\downarrow K & & \downarrow Gh \\
GFA & \xrightarrow{GK} & GB
\end{array}
\]

commute. So we have a bijection.

Naturality in \(B\) is built in:

Given \(FA \xrightarrow{h} B\), we get \(A \xrightarrow{\eta_A} GFA \xrightarrow{Gh} GB\),

\[
\begin{array}{ccc}
FA & \xrightarrow{h} & B \\
\downarrow h' & & \downarrow Gh' \\
FA' & \xrightarrow{Gh'} & GB
\end{array}
\]

Naturality in \(A\) needs \(\eta\) to be a nat. transf. which was built into the def. of \(F\).

Given \(a : A \to GB\), we get \(FA \xrightarrow{h} B\),

\[
\begin{array}{ccc}
A & \xrightarrow{a} & GB \\
\downarrow a' & & \downarrow Gb \\
FA' & \xrightarrow{Gh'} & GB
\end{array}
\]

Satisfying... (overleaf)
Satisfying:

\[
\begin{array}{c}
A \xrightarrow{\eta_A} GFA \xrightarrow{\alpha} GB \\
\end{array}
\]

i.e. both \( h \) and \( h \circ Fa \) are morphisms \((FA, \eta_A) \to (B, \eta'_A) = (B, K_A)\) in \((A \downarrow G)\) so they are the same.

Example \( G \) has limits (resp. colimits) of shape \( J \) iff the functor \( \Delta: G \to [J, G] \) sending \( A \) to the const. diagram \( \Delta A \) has a right (resp. left) adjoint.

Corollary "Uniqueness of adjoints".

Any two left adjoints of a given functor \( G: \mathcal{D} \to \mathcal{C} \) are canonically naturally isomorphic.

**Proof** Assume \( F \) and \( F' \) are both left adjoints of \( G \). Then \((FA, \eta_A)\) and \((FA', \eta'_A)\) are both initial objects of \((A \downarrow G)\). So \( \exists ! \) isomorphism \( \alpha_A: (FA, \eta_A) \to (FA', \eta'_A) \) in \((A \downarrow G)\). Its naturality follows from uniqueness. \( \Rightarrow \alpha \) nat. iso.

Lemma "Adjoints compose". Given \( C \xrightarrow{F} \mathcal{D} \xleftarrow{H} \mathcal{E} \) with \( F \dashv G \) and \( H \dashv K \), then \( HF \dashv GK \).

**Proof** We have bijections \( HFA \to C \), \( \text{natural in } A \) and \( C \) \( \square \).
Category Theory

Corollary "Adjooints in squares".

Let \( \begin{array}{ccc}
E & \rightarrow & B \\
\downarrow G & & \downarrow K \\
F & \rightarrow & D
\end{array} \) be a commutative diagram,

where all of \( F, G, H, K \) have left adj. Then the diagram

\[ \begin{array}{ccc}
E & \leftarrow & B \\
\uparrow & & \uparrow \\
F & \leftarrow & D
\end{array} \]

of left adjoints commutes up to natural isomorphism.

\( \text{Pf} \) Both composites of the square are left adjoints to \( HF = KG \), so they are isomorphic by uniqueness of adjoints. \( \square \)

C. Units and Counits

Def Given an adjunction \( (F \dashv G) \), the nat. transf. \( \eta : 1_G \rightarrow GF \) is called the unit of the adjunction.

Dually, \( \varepsilon : FG \rightarrow 1_F \) is the counit.

Thm "Adjunctions via units and counits"

Given two functors \( \begin{array}{ccc}
E & \rightarrow & D \\
\downarrow G & & \downarrow F
\end{array} \), specifying an adjunction is equivalent to specifying natural transformations \( \eta : 1_E \rightarrow GF \) and \( \varepsilon : FG \rightarrow 1_D \) satisfying the triangular identities. \( \eta \) and \( \varepsilon \) must make the diagrs.

\[ \begin{array}{ccc}
F & \rightarrow & FGF \\
\downarrow F & & \downarrow \varepsilon \\
1_F & \rightarrow & F
\end{array} \quad \text{and} \quad \begin{array}{ccc}
G & \rightarrow & GFG \\
\downarrow G & & \downarrow GE \\
1_G & \rightarrow & G
\end{array} \]

commute.
Proof: Given an adjunction \( F \dashv G \), the unit \( \eta : A \to GFA \) corresponds to \( FA \xrightarrow{\eta_{FA}} FGFA \xrightarrow{\varepsilon_B} FA \) and to \( FA \xrightarrow{1_{FA}} FA \). So the first triangular identity is satisfied: the two maps correspond to the same thing so they must be equal. Dually the second one follows using \( \varepsilon_B \).

Conversely, given \( \eta \) and \( \varepsilon \) satisfying the triangular identities, we must show that the mappings \( f \mapsto G\eta_A \) and \( g \mapsto \varepsilon_B Fg \) are mutually inverse and natural in \( A, B \).

We have commutative diagrams:

\[
\begin{align*}
FA & \xrightarrow{F\eta_A} FGFA \xrightarrow{FGf} FGB \\
\downarrow{1_{FA}} & \downarrow{\eta_{FA}} \downarrow{\varepsilon_B} \\
FA & \downarrow{f} \xrightarrow{B} B
\end{align*}
\]

and

\[
\begin{align*}
A & \xrightarrow{\eta_A} GB \\
\downarrow{\eta_A} & \downarrow{1_{GB}} \\
GFA & \xrightarrow{GFg} GFGB \xrightarrow{G\varepsilon_B} GB
\end{align*}
\]

⇒ since they commute, this proves that the mappings are inverse.

Naturality in \( A, B \) follows from functionality of \( F \) and \( G \) □
Examples (a) Consider $\text{Set} \xleftarrow{G} \text{Gp}$, the "forgetful/free" adjunction. For a set $A$, $\eta_A : A \to G\text{FA}_A$ is the inclusion of the generators, and for a group $B$, $\varepsilon_B : F\text{GB} \to B$ is evaluation.

(b) The abelianisation functor $\text{ab} : \text{Gp} \to \text{AbGp}$ is left adjoint to the inclusion functor $I : \text{AbGp} \to \text{Gp}$.

For a group $G$, $\eta_G : G \to I\text{ab}G = G/[[G,G]]$ is the quotient map.

For an Abelian group, $\varepsilon_A : \text{ab}I\mathbb{A} \to \mathbb{A}$ is the canonical isomorphism $\mathbb{A}/[[\mathbb{A},\mathbb{A}]] \to \mathbb{A}$ (can think of it as identity as $[[\mathbb{A},\mathbb{A}]]$ is trivial).

(c) Consider a space $X$ and the adjunction $PX \xleftarrow{I} \text{EX}$ (cf. Adjunctions example(5)).

The unit is $A \subseteq A$, i.e. any set is inside its closure.

The counit is $F \subseteq F$ (any closed set contains its closure).

(d) Write down unit & counit for all other adjunctions.
Lemma "Reflections":

Given an adjunctions $F \dashv G$ with counit $\varepsilon : FG \to 1_

(i) $G$ is faithful $\iff \varepsilon_B$ is epic for all $B$

(ii) $G$ is fully faithful $\iff \varepsilon_B$ is an iso for all $B$

Proof: (i) Given $g : B \to B'$, $Gg : GB \to GB'$ corresponds under the adjunction to $FGB \xrightarrow{\varepsilon_B} B \xrightarrow{g} B'$

(by nat. of $\varepsilon$). So if $g' : B \to B'$ satisfies $Gg \to Gg'$ and $\varepsilon_B$ is epic, then $g=g'$

so $G$ is faithful.

Conversely, if $G$ is faithful and $g\varepsilon_B = g'\varepsilon_B$

then $Gg = Gg'$ so $g=g'$ and $\varepsilon_B$ is epic.

(ii) Suppose $\varepsilon$ is an iso. By (i) $G$ is faithful.

Given $GB \xrightarrow{\varepsilon_B} GB'$, we can form the composite $G : FGB \xrightarrow{FF} FGB'$

Then $g$ satisfies $FGg = Ff$, so $Gg$

corresponds under the adjunction to $\varepsilon_B$, $FGg = \varepsilon_B Ff$ which is also what $f$

corresponds to, so $Gg = f \implies G$ is full.

Conversely, suppose that $G$ is full and faithful.

We have a morphism $GB \xrightarrow{Gg} GFGB$ which is $Gg$ for a unique $g : B \to FGB$ (existence as $G$ is full, uniqueness as $G$ is faithful).

We show that this is the inverse of $\varepsilon_B$:

we have the triangular identity ...
Category Theory

We have the triangular identity

\[ GB \xrightarrow{\eta_B g} GFGB \]
\[ \xrightarrow{\varepsilon_B g} GB \]
\[ \xrightarrow{\varepsilon_B} GB \]

giving \( \varepsilon_B g = 1_B \) as \( G \) is faithful.

We can also use the other triangular identity and naturality of \( \varepsilon \) to show that \( g \varepsilon_B = 1_{FGB} \)

\[ FGB \xrightarrow{\varepsilon_B} B \]
\[ \xrightarrow{\varepsilon_B \varepsilon_B} FGB \]
\[ \xrightarrow{\varepsilon_B \varepsilon_B} FGB \]

so \( \varepsilon_B \) is an iso.

\[ \square \]

Def: (a) An adjunction with \( G \) fully faithful is called a reflection.

(b) A reflective subcategory is a full subcategory \( \mathcal{D} \) of \( \mathcal{E} \) for which the inclusion functor has a left adjoint.

Examples: (a) We have already seen that \( AbGp \) is reflective in \( Gp \). Given a group \( G \), the commutator subgroup \([G,G]\) has the property that \( G/[G,G] \) is abelian and any homomorphism \( G \rightarrow A \) factors with an abelian factor through \( G \rightarrow G/[G,G] \).
(b) Let \( \mathcal{E} \) be the full subcat of \( \text{AbGp} \) whose objects are torsion groups (those in which every element has finite order). Then \( \mathcal{E} \) is coreflected in abelian groups. (its inclusion factor doesn't have a left adj., but a right adj.).

Given \( A \), the subgroup \( A_t \) of torsion elements in \( A \) is the required coreflection, since any homomorphism \( B \to A \) with \( B \) a torsion group factors through the inclusion \( A_t \to A \).

(c) Let \( \mathcal{E} = \text{Top} \) and \( \mathcal{D} \) be the full subcat of compact Hausdorff spaces. Then the Stone-Čech compactification \( \beta X \) of an arb. space \( X \) is its reflection in \( \mathcal{D} \).
D. Adjoint equivalence

An adjunction whose unit and counit are both isos is in particular an equivalence of categories. We call it an adjoint equivalence.

Lemma "any equivalence can be made into an adjoint one".

Consider an equivalence $\mathcal{E} \xleftarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{D}$, $\alpha: 1_{\mathcal{G}} \xrightarrow{\sim} GF$, $\beta = 1_{\mathcal{D}} \xrightarrow{\sim} FG$. Then there is an adjoint equivalence $(F \dashv G)$ with unit $\alpha$.

Proof We define $E$ as the composite $\mathcal{E} = E: FG \xrightarrow{\beta_{FG}} FGFG \xrightarrow{(FX_0)\beta^{-1}} FG \xrightarrow{\beta^{-1}} 1_{\mathcal{D}}$.

Note here that $\beta_{FG} = FG\beta$, since

$1_{\mathcal{D}} \xrightarrow{\beta} FG$ commutes and $\beta$ is pointwise epic.

Similarly $\alpha_{GF} = GF\alpha$.

We have to verify the triangular identities.

We have:

\[
\begin{align*}
F & \xrightarrow{\beta F} FGF \\
F \times F & \xrightarrow{F \times F} FGF \times FGF \\
GF & \xrightarrow{\beta_{FGF}} FGF \\
F & \xrightarrow{\beta_f} F
\end{align*}
\]

which reduces $F \xrightarrow{F\alpha} FGF \xrightarrow{\alpha_f} F$ to $1_F$. 

Similarly $G \xrightarrow{\varepsilon} GFG \xrightarrow{G\varepsilon} G$ is reduced to 16 □

E. Adjunctions and Limits

Thm "Right adjoints preserve limits".

Suppose $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint. Then $G$ preserves all limits which exists in $\mathcal{D}$.

Proof 1 "apply adjunction to each leg."

Consider a diagram $D : J \to \mathcal{D}$.

Then cones over $GD$ with summit $A$ correspond to cones over $D$ with summit $FA$. Hence, if $D$ has a limit $(L \triangleleft DCj)_j$, each such cone corresponds to a morphism $FA \to L$, which in turn corresponds to a morphism $A \to GL$. So $(GL \xrightarrow{\varepsilon_j} GDCj)$ is a limit cone in $\mathcal{C}$ □

Proof 2 Recall that $\mathcal{D}$ has limits of shape $J$ iff the "constant diagram" functor $\Delta : \mathcal{D} \to [J, \mathcal{D}]$ has a right adj.

So suppose that $\mathcal{C}$ and $\mathcal{D}$ have limits of shape $J$, for some $J$.

Form the commutative square

$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\Downarrow & & \Downarrow \\
[J, \mathcal{C}] & \xrightarrow{\alpha, F} & [J, \mathcal{D}]
\end{array}$

in which all functors have right adjoints.
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So by the "adjoints in squares" lemma, the diagram

\[
\begin{array}{ccc}
\text{[} \mathcal{G}, \mathcal{B} \text{]} & \to & \text{[} \mathcal{G}, \mathcal{C} \text{]} \\
\downarrow \lim_j & & \downarrow \lim_j \\
\mathcal{B} & \to & \mathcal{C}
\end{array}
\]

commutes up to iso, i.e. \( \mathcal{G} \) preserves limits of shape \( \mathcal{G} \).

For a converse to this theorem, we construct initial objects in the cats \((\mathcal{A} \downarrow \mathcal{G})\), under the assumption that \( \mathcal{B} \) has and \( \mathcal{G} \) preserves suitable limits.

**Lemma** "Limits in \((\mathcal{A} \downarrow \mathcal{G})\)"

Consider \( \mathcal{G} : \mathcal{B} \to \mathcal{C} \) and \( A \in \text{ob} \mathcal{B} \). If \( \mathcal{B} \) has and \( \mathcal{G} \) preserves limits of shape \( \mathcal{G} \), then \((\mathcal{A} \downarrow \mathcal{G})\) has limits of shape \( \mathcal{G} \), and the forgetful functor \( U : (\mathcal{A} \downarrow \mathcal{G}) \to \mathcal{B} \) creates them.

**Proof:** Consider a diagram \( D : \mathcal{G} \to (\mathcal{A} \downarrow \mathcal{G}) \).

Write \( D(j) \) as \((UDC(j), f_j)\) where \( f_j : A \to GU D(j) \).

Suppose \((L \xrightarrow{\xi_j} UDC(j))\) is a limit for \( UD \).

Then \((GL \xrightarrow{\xi_j} GU D(j))\) is a limit for \( GUD \) as \( \mathcal{G} \) preserves limits.
But \((f_j)_{j \in G}\) is a cone over GUD, since the edges of GUD lie in \((A \downarrow G)\). So we get a unique morphism \(f : A \to GL\) s.t. \((G \lambda j) f = f j \lambda j\)
i.e. s.t. the \(\lambda j\) become morphisms \((L, f) \to DC(j)\)
in \((A \downarrow G)\)

\[
\begin{array}{c}
GUD(c) \rightarrow GUD(c') \\
\downarrow f_i \quad \downarrow f_j \quad \downarrow G \lambda j \quad \downarrow G \lambda j' \\
A - f - G L
\end{array}
\]

They form a cone over D since \(U\) is faithful (which implies that commutativity of diagrams carries over to cones over D). It is straightforward to verify that this is indeed a limit cone in \((A \downarrow G)\) (check!)

Thm "Primeval Adjoint Functor Theorem".

Suppose \(\mathcal{D}\) has all limits (not just small). Then a functor \(G : \mathcal{D} \to \mathcal{C}\) has a left adjoint \(\iff\) it preserves all limits

Proof "\(\Rightarrow\)" Any right adjoint preserves limits.

"\(\Leftarrow\)" For each \(A \in \text{ob} \mathcal{C}\), \((A \downarrow G)\) has all limits by the "limits in \((A \downarrow G)\)" lemma. So it has an initial object, by the "initial ob. as limit" lemma of §2.D.

So, by the "Adjunctions via initial object" theorem, \(G\) has a left adjoint \(\square\)
However, if a cat \( \mathcal{D} \) has limits for all diagrams over categories "as big as itself", then \( \mathcal{D} \) is a preorder.

The Primeval A.F.T. is useful for posets (cf. ExshQ2) but to get a result applicable to general cats, we need to impose "size restrictions" on \( \mathcal{D} \) and/or \( \mathcal{E} \) to ensure that the "large" limit in the "initial object as limit" lemma can be reduced to a small one.

**Def.** Let \( \mathcal{E} \) be cat. A set of objects \( \{a_i : i \in I\} \) in \( \mathcal{E} \) is called weakly initial if for any \( B \in \text{ob} \mathcal{E} \) there is an \( i \in I \) and a morphism \( h_i : a_i \to B \) in \( \mathcal{E} \).

**Thm. "General Adjoint Functor Theorem."**

Suppose \( \mathcal{D} \) is locally small and complete. Then a functor \( G : \mathcal{D} \to \mathcal{E} \) has a left adjoint \( \mathbb{E} \to G \) preserves all small limits and for each \( A \in \text{ob} \mathcal{E} \), \( (A \downarrow G) \) has a weakly initial set.
Proof: \(\Rightarrow\) \(G\) preserves small limits as a right adjoint, and for each \(A\), 
\(\eta_A : A \to G\) is an initial object of \((A \downarrow G)\), so a one-elem. weakly initial set.

\(\Leftarrow\) By the \(" limits in \((A \downarrow G)\) \" lemma, each \((A \downarrow G)\) is complete; also \((A \downarrow G)\) inherits local smallness from \(\mathcal{D}\). So we just have to show:

Claim: If \(\mathcal{A}\) is complete, locally small and has a weakly initial set, then \(\mathcal{A}\) has an initial object.

Proof of claim: Let \(\{A_j, j \in \text{ob} \mathcal{A}\}\) be the weakly initial set of \(\mathcal{A}\). Form the product

\[ P = \prod_{j \in \text{ob} \mathcal{A}} A_j \]

Then for any \(C \in \text{ob} \mathcal{A}\), there is a morphism \(P \to C\). (\(P\) is a weakly initial object).

Form the diagram \(P \rightrightarrows P \circ \ast\) of all arrows from \(P\) to itself.
Let \(I \to P\) be the limit of \(\ast\) (an "industrial strength equaliser"!)
Note that \(I \rightrightarrows P\) is monic. For any \(C \in \text{ob} \mathcal{A}\) there's a morphism \(I \to C\), namely \(I \rightrightarrows P \to C\). (We want to show it's unique.

Suppose we have \(I \overset{f}{\to} C\). Let \(E \rightrightarrows I\) be their equaliser. As \(E \in \text{ob} \mathcal{A}\), there's \(P \to E\) so the composite \(P \to E \to I \to P\) occurs in \(\ast\)
So \(I \rightrightarrows P\) has equal composite with this and \(I \rightrightarrows P\)
So as \( I \to P \) is monic, we have:
\[
I \to P \to E \to I = 1_I
\]

so \( E \to I \) is a split epi, so \( E \to I \overset{f}{\to} C = E \to I \overset{g}{\to} C \) implies \( f = g \). So \( I \) is an initial object.

\[\square\] (claim)

This proves that \( G \) has a left adjoint using the "Adjunctions via initial objects" theorem.

**Def:** A **coseparating family** \( G \) for a cat \( \mathcal{C} \) is a family of objects \( G = (G_i)_{i \in I} \) s.t. for any pair of morphisms \( A \overset{f}{\to} B \) in \( \mathcal{C} \) with \( f \neq g \), then \( \exists i \in I \) and \( \exists h : B \to G_i \) s.t. \( hf \neq hg \).

**Thm** "Special Adjoint Functor Theorem"

Suppose both \( \mathcal{C} \) and \( \mathcal{D} \) are locally small, and that \( \mathcal{D} \) is complete and well-powered and has a coseparating set.

Then a functor \( G : \mathcal{D} \to \mathcal{C} \) has a left adjoint \( \Leftrightarrow \) \( G \) preserves small limits.

Proof on handout (not examinable)

**Examples** (a) Consider the forgetful functor \( U : \mathcal{Gp} \to \text{Set} \). From the "creating limits" example (a), we know \( \mathcal{Gp} \) has all small limits and \( U \) preserves them; we also know \( \mathcal{Gp} \) is locally small.

To show \( U \) has left adjoint, we need to find a weakly initial set of \( (AVG) \) (so we can use the Gen. A.F.T.)
Given a set $A$, any function $A \rightarrow \mathcal{G}$ factors through $U(H \rightarrow G)$ where $H$ is the subgroup generated by the image of $f$, and $UH$ has cardinality $\leq \max (X_0, \text{card } A)$. But up to isomorphism there is only a set of groups of a given cardinality, and there is only one set of functions from $A$ to any such group.

However, this argument uses most of the machinery required for the explicit construction of free groups.

(b) Consider the inclusion $\mathcal{K}\text{Haus} \rightarrow \text{Top}$. By Tychonoff's theorem, $\mathcal{K}\text{Haus}$ has and $G$ preserves all small products. Similarly for equalisers, since if $X \xrightarrow{g} Y$ is a parallel pair in $\text{Top}$ with $Y$ Hausdorff, then the equaliser $E \rightarrow X$ is a closed subspace of $X$, and so compact if $X$ is.

$\mathcal{K}\text{Haus}$ and $\text{Top}$ are both locally small, and $\mathcal{K}\text{Haus}$ is well-powered, since objects of $X$ correspond to closed subspaces of $X$.

Moreover, $[0,1]$ is a coseparator for $\mathcal{K}\text{Haus}$, by Uryson's lemma. So by the SAFT, $G$ has a left adjoint $\beta$, the Stone-Čech compactification functor.
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4. MONADS

A. Monads and their Algebra

Suppose we have $\mathcal{C} \xrightarrow{F} \mathcal{D}$ with $F \dashv G$. How much of this can we describe without mentioning $\mathcal{D}$?

We have the composite $T = GF: \mathcal{C} \to \mathcal{C}$ and the unit $\eta: 1_\mathcal{C} \to T$, and the nat. transf. $G \varepsilon_F: GFGF \to GF$ which we denote $\mu: TT \to T$.

These satisfy the identities

\[
\begin{array}{c}
T \xrightarrow{T \eta} TT \\
\downarrow 1_T \quad \downarrow \mu \\
T & & T
\end{array}
\quad \text{and} \quad
\begin{array}{c}
T \xrightarrow{\eta T} TT \\
\downarrow 1_T \quad \downarrow \mu \\
T & & T
\end{array}
\]

by the triangular identities of the adjunction, and

\[
\begin{array}{c}
TT \xrightarrow{T \mu} TT \\
\downarrow \mu \quad \downarrow \mu \\
TT & & T
\end{array}
\quad \text{by naturality of } \varepsilon.
\]
Def: A **monad** \( T = (T, \eta, \mu) \) on a cat \( B \) consists of a functor \( T: B \to B \) and nat. trans. \( \eta: 1_B \to T \) (the unit) and \( \mu: TT \to T \) (the multiplication) satisfying the unit laws (1) and (2) and associativity (3).

Example: Given a monoid \( M \), we have a monad structure on the functor \( M \times -: \text{Set} \to \text{Set} \); the unit \( \eta_A: A \to M \times A \) sends \( a \) to \((1, a)\) and the multiplication \( \mu_A: M \times M \times A \to M \times A \) sends \((m, n, a)\) to \( (mn, a) \).

Is this monad induced by an adjunction? Yes. Consider the cat \( M \text{-set} \) of \( M \)-sets; this has a forgetful functor \( G: M \text{-set} \to \text{Set} \) which has a left adjoint \( F \) given by \( FA = M \times A \) with \( M \)-action by multiplication on the left factor. This gives rise to the monad just described.

Def: Let \( T = (T, \eta, \mu) \) be a monad on a cat \( B \). A **\( T \)-algebra** is a pair \((A, \alpha)\) where \( A \in B \) and \( \alpha: TA \to A \) satisfies

\[
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & TA \\
\downarrow{1_A} & & \downarrow{\alpha} \\
A & \xrightarrow{\alpha} & A
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
TTA & \xrightarrow{T\alpha} & TA \\
\downarrow{\mu_A} & & \downarrow{\alpha} \\
TA & \xrightarrow{\alpha} & A
\end{array}
\]

A homomorphism \( f: (A, \alpha) \to (B, \beta) \) of \( T \)-algebras is a morphism \( f: A \to B \) in \( B \) satisfying

\[
\begin{array}{ccc}
TA & \xrightarrow{Tf} & TB \\
\downarrow{\alpha} & & \downarrow{\beta} \\
A & \xrightarrow{f} & B
\end{array}
\]

We write \( B^T \) for the cat of algebras and their homomorphisms.
**Examples:**

(a) The identity functor is a monad on \( \mathcal{E} \), its category of algebras is just \( \mathcal{E} \).

(b) There is a list monad \((\mathcal{L}, \eta, \mu)\) on \( \text{Set} \) as follows.

\[
\mathcal{L} : \text{Set} \to \text{Set} \\
X \mapsto \{ \text{lists } (x_1, \ldots, x_k) \mid k \geq 0, \text{each } x_i \in X \}
\]

and appropriately on morphisms.

\[
\eta_X : X \to \mathcal{L}X \quad \text{"singleton list"} \\
x \mapsto (x)
\]

\[
\mu_X : \mathcal{L}\mathcal{L}X \to \mathcal{L}X \quad \text{is concatenation} \\
((x_1, \ldots, x_m), \ldots, (x_k, \ldots, x_m)) \mapsto (x_1, \ldots, x_1, \ldots, x_k, \ldots, x_m)
\]

An algebra for \( \mathcal{L} \) is a monoid. Indeed, it is a set \( X \) with a map \( \theta : \mathcal{L}X \to X \) giving multiplication.

\[
(\ ) \mapsto e \\
(x_1, \ldots, x_n) \mapsto x_1 \cdot \ldots \cdot x_n
\]

(c) **Powerset monad:** take the covariant powerset functor \( P : \text{Set} \to \text{Set} \).

\[
\eta_X : X \to PX \quad \text{"singleton set"} \\
x \mapsto \{x\}
\]

\[
\mu_X : PPX \to PX \quad \text{is union} \\
\{ A_i \mid i \in I \} \mapsto \bigcup_{i \in I} A_i
\]

An algebra for \( P \) is a complete lattice:

\[
PX \to X \\
A \mapsto \bigvee A \\
X \mapsto T \\
\emptyset \mapsto \bot
\]
Indeed, we get a partial order on $X$: $a \leq b$ if $V[a, b] = b$.

You can check that indeed $a \leq a \leq X$ and $1 \leq a \vee a \leq X$

using diagram (5). As soon as we have all joins and $a \perp 1$, we also get all meets (by the join of the set of lower bounds, which is non-empty as we have $1$).

Algebra homomorphisms are those which preserve arbitrary joins, so the category of algebras is that of sup-complete semi-lattices.

B. Eilenberg-Moore Category

Proposition (Eilenberg-Moore)

There is an adjunction $\mathcal{C} \xleftarrow{\varepsilon} \mathcal{C} \xrightarrow{\eta} \mathcal{C}$ inducing the monad $\eta \eta$.

Proof: We define $G\eta$ as the forgetful functor $(A, \alpha) \mapsto A$, $f \mapsto f$, and $F\eta(A) = (TA, \mu A)$, which is an algebra by (2) and (3). This is called the free $\eta \eta$-algebra.

$F\eta(A \xrightarrow{f} B) = \eta f$ which is a homomorphism by naturality of $\mu$.

Clearly $G\eta F\eta = \eta$, so we take $\eta$ to be the unit of the adjunction. The counit $\varepsilon: F\eta G\eta \rightarrow 1_{\mathcal{C}}$ is defined by $\varepsilon(A, \alpha) = \alpha: (TA, \mu A) \rightarrow (A, \alpha)$ (which is a homomorphism by (5) and natural by (6)).
E. Monadicity

Def: An adjunction $F \dashv G$ is monadic if $K$ is part of an equivalence. We also say $G: D \to C$ is a monadic functor if it has a left adjoint and the adjunction is monadic.

Lemma: Monadic functors reflect isomorphisms

Proof: If $G: D \to C$ is monadic, then exists $F \dashv G$ s.t. $G = G^* K$.

As $K$ is part of an equivalence, it is enough to show that $G^*: C^\text{op} \to C$ reflects isos (for any monad $\tau$).

Given $f: (A, \alpha) \to (B, \beta)$ in $C^\text{op}$ with $f: A \to B$ an iso in $C$, then $f^{-1}$ is also a morphism of algebras:

$$\alpha^* f^{-1} = f^{-1} f \alpha^* T f^{-1} = f^{-1} \beta T f T f^{-1} = f^{-1} \beta$$

So this already tells us that some functions are not monadic.

Example: The forgetful functor $\text{Poset} \to \text{Set}$ does not reflect isos.

\[\begin{array}{ccc}
a & f & b \\
\uparrow & f(c) & \downarrow \\
c & f(b) & \downarrow \\
\end{array}\]

is an iso in $\text{set}$ but not in $\text{Poset}$

But to properly characterise monadic functors we need more. The main idea is that algebras are coequalisers of morphisms between free algebras. Let's make this more precise.
Definition. A reflexive pair in \( \mathcal{C} \) is a parallel pair \( A \xrightarrow{f} B \) for which there exists \( r : B \rightarrow A \) with \( fr = gr = 1_B \).

A reflexive coequaliser is a coequaliser of a reflexive pair.

Definition. A split coequaliser diagram is a diagram of the form
\[
A \xrightarrow{f} B \xleftarrow{g} C
\]
satisfying
\[
h f = h g \quad h s = 1 \quad g t = 1 \quad f t = s h
\]
Recall from ExSh 2 that this makes \( h \) into the coequalizer of \( f \) and \( g \).

Definition. Given a functor \( \mathcal{G} : \mathcal{D} \rightarrow \mathcal{C} \), a parallel pair \( A \xrightarrow{f} B \) in \( \mathcal{D} \) is \( \mathcal{G} \)-split if there exists a split coequaliser diagram in \( \mathcal{C} \)
\[
GA \xrightarrow{Gf} GB \xleftarrow{Gg} C
\]

Examples: (a) Given an adjunction \( \mathcal{G} \xleftarrow{\eta} \mathcal{D} \xrightarrow{\xi} \mathcal{C} \) inducing \((T, \eta, \mu)\) and a \( T \)-algebra \((A, \alpha)\)
\[
\xymatrix{T \mathcal{A} \ar[r]^{T \xi} & T \mathcal{A} \ar[r]^{\alpha} & A}
\]
is a split coequaliser diagram.

So \( \mathcal{F} \mathcal{G} \mathcal{F} \mathcal{A} \xrightarrow{U \xi} \mathcal{F} \mathcal{A} \) is \( \mathcal{G} \)-split.
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(b) Similarly \( GFGFGB \xrightarrow{GF} GFB \xrightarrow{G} GB \)

\( \xrightarrow{\text{is a split coeq diag.}} \)

Lemma "\( \mathbb{T} \)-algebras are coequalisers."

Given a monad \( \mathbb{T} \) on \( \mathcal{C} \) and an algebra \((A, \alpha)\),
the structure map \( \alpha : (TA, \mu_A) \to (CA, \alpha) \) is a
coequaliser in \( \mathcal{C}^\mathbb{T} \).

Proof Consider the diagram

\[
\begin{array}{ccc}
TTA & \xrightarrow{TT\alpha} & TTA \\
\downarrow{\mu_T} & & \downarrow{\mu_A} \\
TTA & \xrightarrow{\mu_A} & TA
\end{array}
\]

\[\begin{array}{ccc}
TTA & \xrightarrow{T\alpha} & TA \\
\downarrow{\mu_T} & & \downarrow{\alpha} \\
TTA & \xrightarrow{\mu_A} & TA
\end{array}\]

in \( \mathcal{C}^\mathbb{T} \)

Here the bottom row is a split coequaliser in \( \mathcal{C} \)
and \( T\alpha \) is split epic.

Given any map \( f : (TA, \mu_A) \to (B, \beta) \) with \( fT\alpha = f\mu_A \),
we get a unique \( g : A \to B \) in \( \mathcal{C} \) satisfying \( g\alpha = f \).
Then as \( T\alpha \) is epic, \( g \) is an algebra homomorphism,
so \((A, \alpha)\) is a coequaliser in \( \mathcal{C}^\mathbb{T} \). \( \square \)

Notice that \((TA, \mu_A) = F^\mathbb{T}G^\mathbb{T}(A, \alpha)\). So the
"primeval" idea behind monadicity theorems is
that we recognise a monadic adjunction by the
fact that for any \( B \in \mathcal{D} \),

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow{G} & & \\
FGFGB & \xrightarrow{E_F \beta} & FGB \xrightarrow{E} B
\end{array}
\]

is a coequaliser.
This diagram is called the standard free presentation of $B$.

**Thm (Precise Monadicity Theorem)**

A functor $G : \mathcal{D} \to \mathcal{C}$ is monadic iff

(i) $G$ has a left adjoint and

(ii) $G$ creates coequalisers of $G$-split pairs.

**Thm (Crude Monadicity Theorem)**

Consider $G : \mathcal{D} \to \mathcal{C}$ s.t.

(i) $G$ has a left adjoint

(ii) $G$ reflects isomorphisms

(iii) $\mathcal{D}$ has and $G$ preserves reflexive coequalisers

then $G$ is monadic.

**Proof (Precise "$\Rightarrow$")** (i) If $G$ is monadic, then $G$ has a left adjoint by definition.

(ii) Sufficient to show that the forgetful functor $G^\triangleright : \mathcal{C}^\triangleright \to \mathcal{C}$ creates coequalisers of $G^\triangleright$-split pairs. If $(A, \alpha) \xrightarrow{g} (B, \beta)$ is a parallel pair in $\mathcal{C}^\triangleright$ and $A \xrightarrow{f} B \xrightarrow{h} C$ is a split coeq. in $\mathcal{C}$.

Then $TA \xrightarrow{Tf} TB \xrightarrow{Th} TC$ is also a coequaliser. So as $h\beta T \circ f = h \circ \alpha \circ g \circ \alpha = h \circ \beta Tg$ we get a unique $\gamma : TC \to C$ s.t. $TB \xrightarrow{Th} TC$ commutes.

\[ \begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{h} & C \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
\gamma \circ Tg & & \gamma \circ Tg & & \gamma \circ Tg
\end{array} \]
To show that \((C, \xi)\) is a \(T\)-algebra, i.e. that 
\[ \eta c = 1c \quad \text{and} \quad \xi TX = \mu c, \]
it is enough to show that 
\[ \eta c h = h \quad \text{and} \quad \xi TX TTh = \mu c TTh, \]
as \(h\) and \(TTh\) are coequalisers.

These two equations follow from nat. of \(\eta\) and \(\mu\), \(\ast\) and the fact that \((B, \beta)\) is a \(T\)-algebra.

Then \(h : (B, \beta) \rightarrow (C, \xi)\) is the coequaliser of \(f\) and \(g\) in \(\mathcal{E}^T\) (proof as in previous lemma). \(\square\)

Proof (Precise "\(\Rightarrow\)" and Crude)

We have \(\xymatrix{ B \ar[r]^k & \mathcal{E}^T \ar@/^/[l]_F \ar[r]^G & \mathcal{E}}\). We will construct a left adjoint \(L : \mathcal{E}^T \rightarrow \mathcal{E}\) for \(K\) and the unit and counit of \(L \rightarrow K\) and show that they are isos.